Error of an eFDDM: What do matched asymptotic expansions teach us?

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1 Introduction

In this paper, we are interested in heterogeneous decomposition methods. For complex problems, it may be useful to rely on approximations on subdomains and obtain an approximate global solution through appropriate coupling conditions on the interface. For an overview of such techniques, see [5] and references therein. In particular, we want to look at methods that neglect diffusion in a subdomain of non-zero measure. Gander and Martin [6] have compared the existing coupling methods with respect to their order in the small parameter in the different subdomains. An example of such a method is the χ-method, see [2, 1]. We want to extend these results to the Fuzzy Domain Decomposition Methods developed by Gander and Michaud [7]. This method is interesting as it provides an adaptive coupling method that allows for a tracking of domain of validity of different approximations. In [7], the authors show an approximation error analysis for a very simple problem that does not seem to generalize to higher dimensions. We develop a more general analysis based on matched asymptotic expansions [3] that show the convergence of an explicit FDDM (eFDDM) [7] method. For the comparison with the result of Gander and Martin [6], we note that our results compare with their $a < 0$ case. They show that the coupling is usually of order $\mathcal{O}(\nu)$, unless a factorization of the operator is done, in which case, the result can be improved to get an order of $\mathcal{O}(\nu^m)$. We show that an eFDDM is of order $\mathcal{O}(\nu)$ and have numerical evidence that (in 1D at least) this method is of order $\mathcal{O}(\nu^{3/2})$ in the subdomain where diffusion is taken into account.

Basic facts about eFDDMs: Following [7], we recall that an eFDDM is a numerical method based on a FDD $\Omega = \Omega_1 + \cdots + \Omega_n$, where Ω_i are fuzzy sets of membership functions h_i and $\sum_{i=1}^{n} h_i = 1$. In this paper, we

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will work with a FDD of two subdomains Ω_1 and Ω_2 of membership function $h_1 = h$ and $h_2 = 1 - h$ respectively.

We approximate the linear problem with zero Dirichlet boundary condition

$$\mathcal{L}(u) = f \quad \text{on } \Omega, \quad u|_{\partial\Omega} = 0, \tag{1}$$

using two approximations $\mathcal{L}_i$, $i = 1, 2$, valid in a fuzzy sense in Ω_i.

We have the *global approximation*

$$h\mathcal{L}_1(u) + (1-h)\mathcal{L}_2(u) = f, \quad \text{on } \Omega, \quad u|_{\partial\Omega} = 0, \tag{2}$$

equivalent to the *eFDD approximation*

$$\begin{cases} \widetilde{\mathcal{L}}_1(u_1) = hf + \mathcal{L}_{12}(u_2) & \text{on Supp}(\Omega_1), \quad u_i|_{\partial\Omega} = 0, \\ \widetilde{\mathcal{L}}_2(u_2) = (1-h)f + \mathcal{L}_{21}(u_1) & \text{on Supp}(\Omega_2), \end{cases} \tag{3}$$

with $u_i = h_i u$ and $\widetilde{\mathcal{L}}_i$ and $\mathcal{L}_{ij}$ linear operators coming from the application of the product rule to exchange h with the operators $\mathcal{L}_i$, see [7] for details.

2 Model problem

We are interested in the reaction diffusion model problem

$$\begin{cases} \mathcal{L}_{h\nu}(u_{h\nu}) := -h\nu\Delta u_{h\nu} - a\cdot\nabla u_{h\nu} + cu_{h\nu} = f, & \text{in } \Omega \\ u_{h\nu} = 0, & \text{on } \partial\Omega \end{cases} \tag{4}$$

where $\nu > 0$, $a > 0$ and $c(x) + \text{div } a(x)/2 - \nu\Delta h/2 \geq \alpha > 0$ a.e. in a smooth domain Ω, $0 \leq h \leq 1$ is a smooth function with $\nabla(h^{1/2}) \in L^2(\Omega)^1$.

We want to study the approximation error of an eFDDM for an approximation of $\mathcal{L}_{1\nu}(u_{1\nu}) = \mathcal{L}_\nu(u_\nu) = f$ by the global approximation $h\mathcal{L}_\nu(u) + (1-h)\mathcal{L}_{0\nu}(u) = \mathcal{L}_{h\nu}(u_{h\nu}) = f$, which can be written in the eFDDM as in (3). We multiply (4) by $v \in H_0^h = \{u \in L^2(\Omega), h^{1/2}\nabla u \in L^2(\Omega), (h^{1/2}u)|_{\partial\Omega} = 0\}$ (this is a Hilbert space for the inner product $(u,v)_{L^2} + (h^{1/2}\nabla u, h^{1/2}\nabla v)_{L^2}$) and integrate by parts to obtain the following variational formulation

$$\begin{cases} \text{Find } u_{h\nu} := \tilde{u} \in H_0^h \text{ such that for every } v \in H_0^h, \\ a_{h\nu}(\tilde{u}, v) := \nu\int_\Omega h\nabla\tilde{u}\cdot\nabla v dx - \int_\Omega[((a-\nu\nabla h)\cdot\nabla\tilde{u})v + c\tilde{u}v]dx = \int_\Omega fv dx. \end{cases} \tag{5}$$

In order to see that problem (5) is well-posed, we need the following lemma.

Lemma 1. *If $c(x) + \text{div } a(x)/2 - \nu\Delta h/2 \geq \alpha > 0$ a.e., where α is independent of ν, we have:*

[1] This is only a technicality to guaranty the wellposedness of the trace $h^{1/2}u$ on $\partial\Omega$. Typical smooth "plateau" functions satisfy this condition.

$$a_{h\nu}(u,u) \geq \nu \|h^{\frac{1}{2}}\nabla u\|^2_{L^2(\Omega)} + \alpha\|u\|^2_{L^2(\Omega)}, \tag{6}$$

$$\|u_{h\nu}\|_{L^2(\Omega)} \leq \frac{1}{\alpha}\|f\|_{L^2(\Omega)}. \tag{7}$$

Proof. In order to obtain a lower bound of the bilinear form we use

$$\begin{aligned} a_{h\nu}(u,u) &= \nu\|h^{\frac{1}{2}}\nabla u\|^2_{L^2(\Omega)} + \int_\Omega (c+\frac{1}{2}\text{div}a)|u|^2dx - \frac{\nu}{2}\int_\Omega \Delta h|u|^2dx \\ &\geq \nu\|h^{\frac{1}{2}}\nabla u\|^2_{L^2(\Omega)} + \alpha\|u\|^2_{L^2(\Omega)}. \end{aligned} \tag{8}$$

The first equality follows from the definition of the bilinear form using an integration by parts and the divergence theorem to rewrite $\int_\Omega u(a\cdot\nabla u)dx = -\frac{1}{2}\int_\Omega(\text{div}a)|u|^2dx$.

The *a priori* estimate (7) follows from the fact that $a_{h\nu}(u_{h\nu},u_{h\nu}) \leq \|u_{h\nu}\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)}$ and using (6). □

Remark 1. We want the constant $\alpha > 0$ to be independent on ν. In general, this induces a restriction on h since $\nu\Delta h/2$ needs to be small. For example this is achieved if h is independent of ν.

We assume that (4) has a solution in H^h_0 at least, then the *a priori* estimate (7) ensures the uniqueness and the stability of the solution whenever the assumptions of Lem. 1 holds.

3 Matched asymptotic expansion

From now on, we restrict ourself to a 1D problem with constant coefficient on $\Omega = (0,1)$. We want to use matched asymptotic expansions to study of the approximation error of the eFDDM. Therefore we compute a matched asymptotic expansions solution of (4) assuming that the membership function $h = 1$ at least in the boundary layer of size of order ν forming near 0 [3].

To obtain a matched asymptotic expansions solution, we use:

1. For the external field we assume that $u(x) \approx \sum_{k\geq 0}\nu^k\varphi_k(x),\ x\in(0,1]$.
2. For the internal field, we zoom in the boundary layer by rescaling x. This is done by setting $X = x/\nu$ and assuming that $u(\nu X) = \Phi(X) \approx \sum_{k\geq 0}\Phi_k(X)\nu^k$.

The zeroth-order approximation, to which we will restrict our analysis, is obtained by solving the following system [3]

$$\begin{cases} -a\varphi_0' + c\varphi_0 = f, \quad \varphi_0(1) = 0, \\ -\Phi_0'' - a\Phi_0' = f(0), \quad \Phi_0(0) = 0, \\ \lim_{X\to\infty}\Phi_0(X) = \varphi_0(0). \end{cases} \tag{9}$$

If $f(0) = 0$, the solution of this system is given by

$$\Phi_0(X) = \frac{1-e^{-aX}}{a}\int_0^1 f(y)e^{-\frac{cy}{a}}dy, \quad \varphi_0\left(x\right) = \frac{1}{a}\mathrm{e}^{\frac{cx}{a}}\int_x^1 f\left(y\right)\mathrm{e}^{-\frac{cy}{a}}dy; \quad (10)$$

otherwise the matching fails and the system does not have any solution.

We obtain a globally valid approximation by merging the two solutions using a partition of unity $\{\chi, 1-\chi\}$

$$\tilde{u}_{\nu,\chi}(x) := \chi(x)\Phi_0(\frac{x}{\nu}) + (1-\chi(x))\varphi_0(x); \quad (11)$$

$$\chi(x) := \begin{cases} 1, & \text{if } x < d_1\nu^s \\ \chi^* \in [0,1], & \text{if } d_1\nu^s \le x \le d_2\nu^s, \\ 0, & \text{otherwise} \end{cases} \qquad 0 < s < 1, \quad (12)$$

is smooth. Note that if we scale the χ function $\chi(x\nu^s)$, then χ and its derivatives become independent of ν.

Lemma 2. *For every function χ defined as in* (12)*, we have*

$$\|\chi^{(n)}\|_{L^\infty(\Omega)} = \mathcal{O}(\nu^{-ns}). \quad (13)$$

Proof. This result is a direct consequence of the independence of $\chi(x\nu^s)$ on ν. We change the variable in the function χ and every derivative leads to an additional factor of ν^{-s}, hence the result. □

4 Approximation error estimates

We use a membership function similar to χ to simplify the computations

$$h(x) := \begin{cases} 1, & \text{if } x < c_1\nu^t \\ h^*(x) \in [0,1], & \text{if } c_1\nu^t \le x \le c_2\nu^t \\ 0, & \text{otherwise,} \end{cases} \quad (14)$$

and have the following result:

Theorem 1. *Let $u_{h\nu}$ be the weak solution of* (4) *with constant $a \neq 0$ and c, h defined in* (14) *with $0 \le t < 1$ such that $c - \nu\Delta h/2 \ge \alpha > 0$ a.e. and $\tilde{u}_{\nu,\chi}$ the globally valid approximation of the corresponding first term in the matched asymptotic expansions. Assume also that $f(0) = 0$ and $f \in W^{1,\infty}(\Omega)$. For $\Omega = (0,1)$ and $s = 2/3 + t/3$ in* (12)*, we have*

$$\|u_{h\nu} - \tilde{u}_{\nu,\chi}\|_{L^2(\Omega)} = \mathcal{O}(\nu^{1+t/2}). \quad (15)$$

Proof. We look at the equation for the error and use the fact that the internal and external fields satisfy (9) and $\mathcal{L}_{h\nu}(\tilde{u}_{\nu,\chi}) = (-h\Delta - a\cdot\nabla + c)(\chi\Phi_0 + (1-\chi)\varphi_0)$. The triangle inequality implies

$$
\begin{aligned}
&\|\mathcal{L}_{h\nu}(u_{h\nu}-\tilde{u}_{\nu,\chi})\|_{L^2(\Omega)} = \|f-\mathcal{L}_{h\nu}(\tilde{u}_{\nu,\chi})\|_{L^2(\Omega)} \\
&\le \|\chi f\|_{L^2(\Omega)} + \left\|c\chi\Phi_0(\frac{\cdot}{\nu})\right\|_{L^2(\Omega)} + \left\|(\Phi_0(\frac{\cdot}{\nu})-\varphi_0(\cdot))(h\nu\chi''+a\chi')\right\|_{L^2(\Omega)} \\
&\quad + \left\|2h\nu\chi'(\Phi_0'(\frac{\cdot}{\nu})-\varphi_0'(\cdot))\right\|_{L^2(\Omega)} + \left\|\nu(1-h)\chi(\Phi_0''(\frac{\cdot}{\nu})-\varphi_0''(\cdot))\right\|_{L^2(\Omega)} \\
&\quad + \|\nu(h-\chi)\varphi_0''\|_{L^2(\Omega)} \\
&\le \|f\|_{L^2(0,d_2\nu^s)} + c\left\|\Phi_0(\frac{\cdot}{\nu})\right\|_{L^2(0,d_2\nu^s)} \\
&\quad + \left\|(\Phi_0(\frac{\cdot}{\nu})-\varphi_0(\cdot))\right\|_{L^2(d_1\nu^s,d_2\nu^s)} \|(\nu\chi''+a\chi')\|_{L^\infty(d_1\nu^s,d_2\nu^s)} \\
&\quad + 2\nu\|\chi'\|_{L^\infty(d_1\nu^s,d_2\nu^s)}\left\|(\Phi_0'(\frac{\cdot}{\nu})-\varphi_0'(\cdot))\right\|_{L^2(d_1\nu^s,d_2\nu^s)} \\
&\quad + \nu\left\|(\Phi_0''(\frac{\cdot}{\nu})-\varphi_0''(\cdot))\right\|_{L^2(0,d_2\nu^s)} + \nu\|\varphi_0''\|_{L^2(d_1\nu^s,c_2\nu^t)}.
\end{aligned} \tag{16}
$$

The second inequality follows from the definition of χ using the support of its derivatives. In order to finish the proof, we need a technical lemma.

Lemma 3. *Let $s<1$, $\Omega^s=(\kappa_1\nu^s,\kappa_2\nu^s)$ and $f(0)=0$. For $n=0,1,2$ we have the following estimates*

$$
\left\|\frac{d^n}{dx^n}\left(\Phi_0(\frac{\cdot}{\nu})-\varphi_0(\cdot)\right)\right\|_{L^2(\Omega^s)} = \mathcal{O}(\nu^{\frac{5}{2}s-ns}). \tag{17}
$$

Proof. We start by computing the derivatives of $\Phi_0(\frac{x}{\nu})-\varphi_0(x)$:

$$
\Phi_0(\frac{x}{\nu})-\varphi_0(x) = \frac{1}{a}\left[\int_0^x f(y)e^{-\frac{cy}{a}}dy - e^{-\frac{ax}{\nu}}\int_0^1 f(y)e^{-\frac{cy}{a}}dy\right],
$$

$$
\frac{d}{dx}\left(\Phi_0(\frac{x}{\nu})-\varphi_0(x)\right) = \frac{1}{a}f(x)e^{-\frac{cx}{a}} + \frac{1}{\nu}e^{-\frac{ax}{\nu}}\int_0^1 f(y)e^{-\frac{cy}{a}}dy,
$$

$$
\frac{d^2}{dx^2}\left(\Phi_0(\frac{x}{\nu})-\varphi_0(x)\right) = \left(\frac{f'(x)}{a}-\frac{cf(x)}{a^2}\right)e^{-\frac{cx}{a}} - \frac{a}{\nu^2}e^{-\frac{ax}{\nu}}\int_0^1 f(y)e^{-\frac{cy}{a}}dy.
$$

In order to estimate the L^2-norm of these expressions, we use the fact that $\|\int_0^x f(y)dy\|_{L^2(\Omega^s)} \le \sqrt{3}\nu^{s/2}(\kappa_2^3-\kappa_1^3)^{1/2}\|f\|_{L^\infty(\Omega^s)}/3$, $\|f\|_{L^2(\Omega^s)} \le \nu^{s/2}(\kappa_2-\kappa_1)^{1/2}\|f\|_{L^\infty(\Omega^s)}$ and the fact that $e^{-\frac{cx}{a}}<1$, for all $x\in(0,1)$. Furthermore, as $f(0)=0$, we have $\|f\|_{L^\infty(\Omega^s)} \le \nu^s\kappa_2\|f'\|_{L^\infty(0,\kappa_2)}$, hence we have

$$
\begin{aligned}
\left\|\Phi_0(\frac{\cdot}{\nu})-\varphi_0(\cdot)\right\|_{L^2(\Omega^s)} &\le \frac{\sqrt{3}\|f\|_{L^\infty(\Omega^s)}}{3a}\nu^{\frac{3s}{2}}(\kappa_2^3-\kappa_1^3)^{\frac{1}{2}} + \mathcal{O}(\nu^{\frac{1}{2}}e^{\frac{-a\kappa_1}{\nu^{1-s}}}) \\
&\le C_1\nu^s\|f'\|_{L^\infty(\Omega^s)}\nu^{\frac{3s}{2}} + \mathcal{O}(\nu^{\frac{1}{2}}e^{\frac{-a\kappa_1}{\nu^{1-s}}}) \\
&= \mathcal{O}(\nu^{5s/2}+\nu^{\frac{1}{2}}e^{\frac{-a\kappa_1}{\nu^{1-s}}}),
\end{aligned}
$$

$$\begin{aligned}\left\|\frac{d}{dx}\left(\Phi_0(\frac{\cdot}{\nu})-\varphi_0(\cdot)\right)\right\|_{L^2(\Omega^s)} &\leq \frac{\sqrt{\kappa_2-\kappa_1}}{a}\nu^{s/2}\|f\|_{L^\infty(\Omega^s)}+\mathcal{O}(\nu^{-\frac{1}{2}}e^{\frac{-a\kappa_1}{\nu^{1-s}}})\\ &\leq C_2\nu^s\|f'\|_{L^\infty(\Omega^s)}\nu^{\frac{s}{2}}+\mathcal{O}(\nu^{-\frac{1}{2}}e^{\frac{-a\kappa_1}{\nu^{1-s}}})\\ &= \mathcal{O}(\nu^{3s/2}+\nu^{-\frac{1}{2}}e^{\frac{-a\kappa_1}{\nu^{1-s}}}),\end{aligned}$$

$$\begin{aligned}\left\|\frac{d^2}{dx^2}\left(\Phi_0(\frac{\cdot}{\nu})-\varphi_0(\cdot)\right)\right\|_{L^2(\Omega^s)} &\leq \sqrt{\kappa_2-\kappa_1}\nu^{s/2}\left\|\left(\frac{f'}{a}-\frac{cf}{a^2}\right)\right\|_{L^\infty(\Omega^s)}+\mathcal{O}(\nu^{-\frac{3}{2}}e^{\frac{-a\kappa_1}{\nu^{1-s}}})\\ &= \mathcal{O}(\nu^{s/2}+\nu^{-\frac{3}{2}}e^{\frac{-a\kappa_1}{\nu^{1-s}}}).\end{aligned}$$

We obtain the desired result noting that if $s<1$ then the exponential terms are negligeable and can be neglected in the $\mathcal{O}$. □

We can now finish the proof of Thm. 1. Using Eqs. (13) and (17) and estimates previously used for the norms of f and φ_0''. Eq. (16) becomes

$$\begin{aligned}\|f-\mathcal{L}_{h\nu}(\tilde{u}_{\nu,\chi})\|_{L^2(\Omega)} &= \mathcal{O}(\nu^{3s/2})+\mathcal{O}(\nu^{3s/2})+\mathcal{O}(\nu^{5s/2})\left(\mathcal{O}(\nu^{1-2s})+\mathcal{O}(\nu^{-s})\right)\\ &\quad+\nu\mathcal{O}(\nu^{-s})\mathcal{O}(\nu^{3s/2})+\nu\mathcal{O}(\nu^{s/2})+\nu\mathcal{O}(\nu^{t/2})\\ &= \mathcal{O}(\nu^{3s/2}+\nu^{1+s/2}+\nu^{1+t/2})\end{aligned}$$

We know that $t<s$ by hypothesis so that the second term is subdominant, choosing s such that $3s/2=1+t/2$ gives the condition on s in Thm. 1. We conclude the proof using the *a priori* estimate (7). □

Corollary 1. *The approximation error done by the use of an eFDDM as described in Sect. 2 is of order 1 in ν, that is*

$$\|u_\nu-u_{h\nu}\|_{L^2(\Omega)}=\mathcal{O}(\nu). \tag{18}$$

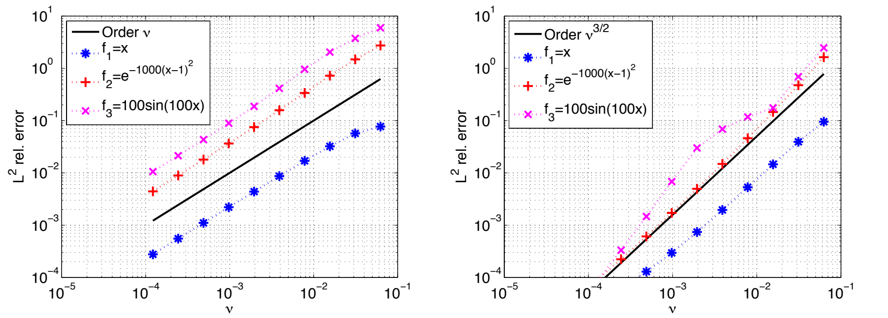
Proof. This follows from Thm. 1 by the triangle inequality, noting that $h=1$ implies $t=0$. □

The approximation error obtained here is global. We now show a numerical example that illustrates the local convergence of the approximation error of the method.

Numerical experiment: We show here that an eFDDM for the problem $\mathcal{L}_\nu(u)=f$ on $\Omega=(0,1)$ is of order $\mathcal{O}(\nu)$ as predicted by Corol. 1 and that it is numerically of order $\mathcal{O}(\nu^{3/2})$ in the subdomain where diffusion is taken into account. For this, we solve the corresponding eFDD approximation (3) with $\mathcal{L}_1:=\mathcal{L}_\nu$ and $\mathcal{L}_2:=\mathcal{L}_{0\nu}$ and $a=c=1$, see [7] for the definition of the operators $\widetilde{\mathcal{L}}_k$ and $\mathcal{L}_{kl}$, $k,l=1,2$.

We define h as in (14) with h^* a cubic spline on $(c_1\nu^t,c_2\nu^t)$,

$$h^*(x)=\delta^{-3}(2x^3-3\nu^t(c_1+c_2)x^2+6\nu^{2t}c_1c_2x-c_2^2\nu^{3t}(3c_1-c_2)),$$



(a) Results for the pure advective subdomain. Approximation error of order 1.

(b) Results for the diffusive subdomain. Approximation error of order 3/2.

Fig. 1 Approximation errors where we refined the grid keeping $n\nu$ constant.

with $\delta := (c_2 - c_1)\nu^t$ and $0 < c_1\nu^t \le c_2\nu^t \le 1$.

In order to satisfy the hypothesis of Thm. 1, we need to have $\alpha > 0$. In our case, we have $\|h''\|_{L^\infty(\Omega)} = 6/\delta^2$ which implies the condition $\nu^{t-1/2} > (3/c)^{1/2}/(c_2 - c_1)$. Choosing $t = 1/2$, $c_1 = 6$ and $c_2 = 8$, we satisfy this condition and we expect an order of convergence of $\mathcal{O}(\nu^{5/4})$ in the diffusive domain. Intuitively we can understand this result by Thm. 1, as both u_ν and $u_{h\nu}$ have $h = 1$ in this domain. A triangle inequality then implies the result. This order of convergence is better than the order of most of the available methods [6], but not optimal. Using the same reasoning, we can hope for a $\mathcal{O}(\nu^{3/2-\varepsilon})$ for $t = 1 - \varepsilon$.

We now show an numerical example with $t = 0.99$ that realizes an order $\mathcal{O}(\nu^{3/2-\varepsilon})$. Even if we can not prove the corresponding hypothesis in this case, the numerical example behaves as expected.

We introduce a set of equidistant points $x_i = i \cdot \Delta x$, $i = 0, \ldots, n+1$ and $\Delta x = 1/(n+1)$ and discretize the eFDDM with an upwind 3-point finite difference scheme. This gives us a system of $2n$ coupled equations. For each component u_j, $j = 1, 2$, we remove from the system all the irrelevant equations, those for which $h_j(x_i) = 0$; this corresponds to the restriction to $\mathrm{Supp}(\Omega_j)$. In order to obtain an approximation error curve, we let ν tends to 0 keeping $n\nu$ constant to insure the resolution of the boundary layer. This is just to test the behavior of the method. In Figure 1 we display the L^2 relative error between the numerical approximations of u_ν and $u_{h\nu}$ computed with the eFDDM scheme for three choices of f.

We see that for the three choices of f the method behaves as predicted by Corol. 1, that is the error is of order $\mathcal{O}(\nu)$ in the advective subdomain. And we see numerically that the error curves are of order $\mathcal{O}(\nu^{3/2})$ in the diffusive subdomain, as expected.

5 Conclusion

In this paper we have shown that matched asymptotic expansions are useful for the analytical study of approximation error of an eFDDM. We have proved that the error is of order ν by taking advantage of the similarities between the two approaches. The first is based on a decomposition of the operator whereas the second is based on a decomposition of the solution.

Our results compare the results for $a < 0$ in Gander and Martin [6] with Dirichlet boundary conditions. We have proven that an eFDDM is not worse than the other coupling methods and our numerical example shows that we are in fact better inside the diffusive subdomain. The justification of the order $\mathcal{O}(\nu^{3/2})$ in the diffusive subdomain is only heuristic as we have not been able to prove it yet. We will address this problem and get local estimates in future work. The only other known method that achieves an order better than $\mathcal{O}(\nu)$ is the one based on the factorization of the operator, which does not generalize to higher dimensions. Our method generalize to higher dimensions and we are working on extension of this work to 2D, 3D and time-dependent problems. We also want to generalize the method to more complicated problem such as the kinetic equations. This has been done for example in the work of Degond et al. [4], but without any approximation error analysis.

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