ON THE EXISTENCE AND UNIQUENESS OF SOLUTION FOR SOME FREQUENCY-DEPENDENT PARTIAL DIFFERENTIAL EQUATIONS COMING FROM THE MODELLING OF METAMATERIALS *

PIERRE-HENRI COCQUET†, PIERRE-ALAIN MAZET† , AND VINCENT MOUYSSET†

Abstract. Several systems coming from the theory of linear wave propagation are investigated, on a bounded domain, in presence of frequency-dependent materials like metamaterials. For each system we show generic well-posedness results under assumptions that are relevant for some models of the literature. This means existence and uniqueness of solution for all frequency except for a discrete locally finite and possibly empty set of frequencies. Finally, some examples of materials are studied like a periodic array of Split-Ring-Resonators (SRR), a chiral metamaterial based on the Ω-particle resonator model, a bi-anisotropic metamaterial made from SRR, absorbing boundary conditions of Perfectly-Matched-Layers (PML) type for the acoustics waves, an example of acoustic metamaterial having negative bulk modulus and an elastic metamaterial.

Key words. Metamaterials, Maxwell's equations, Wave equation, linear elasticity.

AMS subject classifications. 35Q61, 35L05, 35Q74, 78A25, 78A48.

1. Introduction. In 1968, V.G Veselago [34] theoretically investigated the effects of electromagnetic and acoustic phenomena in materials having simultaneously negative values for the permittivity ε and the permeability μ. As they reverse the Poynting vector, they were called "Left-Handed-Materials" (LHM). V.G. Veselago also noticed that they have exotic properties like a reversed Doppler effect or a reversed Vavilov-Cerenkov radiation effect. The keen interest for the LHM was initiated by J.B Pendry in 2000 [29, 30]. He managed to build LHM using a periodic array of Split-Ring-Resonators (SRR). Thus these exotic structures were named "metamaterials". There are various planned applications of metamaterials such as the perfect lens [29], sound focusing [16], cloaking effect using the concept of transformation optics [5, 14] or controlling light using photonic crystals [28].

However, as metamaterials cannot be found in nature as negative indexes materials, they are usually seen as the result of a frequency dependent homogenization (see for example [11, 17, 32, 33]). The homogenized parameters which may depend on the pulsation w become negative definite for some w. However, the usual framework to prove the well-posedness of the systems modelling electromagnetics, acoustics or elastodynamics wave propagation in materials assumes the parameters to be positive definite. The question we thus want to ask is: What happen for the existence and uniqueness of solutions to the equations mentioned above in presence of metamaterials?

Another way to treat these problems should be to consider minimization variational principles like those studied in [25] for the acoustics, elastodynamics and electromagnetism in lossy inhomogeneous bodies at fixed frequency. These principles rely on the introduction of a saddle point minimization problem in order to study, for instance, Maxwell's equation. However, this paper does not give existence and uniqueness of solution to Maxwell's equation. Moreover, to be applied, a special decomposition of the physical parameters have to be achieve in order to get some

*Third revised version of the paper

†ONERA-The french aerospace lab, F-31055, Toulouse, France (Pierre-Henri.Cocquet@onera.fr, Pierre.Mazet@onera.fr and Vincent.Mouysset@onera.fr)

positive definite tensors that we were not able to exhibit in the case of negative index materials. An additional way to deal with the well-posedness of partial differential equations in presence of metamaterials should be to use some fixed-point methods like those introduced in [7] wherein Maxwell's equation in presence of some chiral media are solved. However the development of this method is only led in case of variable but positive parameters and thus does not extend to the case of metamaterials. Consequently, we are not able to use such methods to study existence and uniqueness of solution for systems of partial differential equation with sign-changing coefficients.

On the subject of the mathematical study of problems involving metamaterials, there exists, to the best of author knowledge, only few works. For scalar equations in presence of media having sign-shifting coefficients, some studies were done in [2, 3]. To be more precise, these papers study a second-order scalar problem with parameters that does not depend on the frequency, for the transmission problem between a "classical" material and a "negative" one. Then, using an integral equation technique on the interface between the two media, they manage to prove that the considered problem can be solved with Fredholm alternative and give a new way to approximate the solution of these kind of problems [1]. In [3], the authors use the so-called T-coercivity (or inf-sup) approach to show that if the interface between the "classical" material and the metamaterial is smooth then the problem rely once again on the Fredholm alternative. They can even build in some particular geometric setting the operator T leading to the existence and uniqueness of solution. For works dealing with second order vector equations, we can cite [12] were Maxwell's equation in presence of bianisotropic material and metamaterials are studied. Actually they give conditions on the materials that allows the use of Lax-Milgram theorem and prove the convergence of the classical finite element method within this framework.

However, from a remark of V.G Veselago [34], the parameters of a metamaterial have to depend on the pulsation w and could become negative definite only for some w in a bounded set. The motivation of a frequency-dependence for the homogenized metamaterials comes also from the ability to describe a material coming from a well-posed system on time-domain from an energy point of view. Furthermore, one can notice (see for instance [21, 32, 35]) that the homogenized coefficients of such materials usually depend on the frequency like rational fractions. As a result they can be considered as holomorphic function of $p = iw + \delta$ on some connected open set of $\mathbb{C}$. Actually, it is worth noting that these observations have been confirmed by experiments performed on several metamaterials obtained from a homogenization process [17, 21, 32, 33, 10, 36, 23]. Remark that the previously mentioned well-posedness results are established for metamaterials that does not depend on the frequency. Nevertheless, these results can be used to study frequency-dependent metamaterials having specific physical parameters for $w = w_0$. However the main tool to obtain well-posedness of frequency-dependent partial differential equation is the Fredholm analytical theory (see [19] p.371) which discards an unknown set of frequencies which is not *a priori* empty and hard to determine (see for instance [22]). Thus we cannot state *a priori* the existence and uniqueness of solution for frequency-dependent metamaterials at w_0, whatever the geometry is, since it could happen that w_0 has been discarded away. Consequently we cannot use the previously mentioned results to straightforwardly study the examples mentioned above.

In this paper, we are going to introduce a frequency dependence of the physical parameters invoked in first order symmetric systems describing wave propagation through electromagnetic, acoustic or linear elastic metamaterials. More precisely, a global rather than local approach in the frequency domain is considered and our goal is to provide well-posedness for the frequency-dependent cases mentioned before. Unfortunately, we will not be able to prove such results for all frequencies. Instead we formulate generic well-posedness that is existence and uniqueness except for a discrete and locally finite set of frequencies. The key tool for the proof of the presented results is the Fredholm theory which implies to remove some frequencies depending on what the geometry and the boundary conditions are. Consequently, our results are not global well-posedness. Nevertheless, we are going to give here sufficient and general conditions on the materials leading to discreteness and local finiteness for the singular frequencies to be dropped out.

An outline of this article is as follows. First of all, a mathematical framework to study Maxwell's equation, acoustic wave system and linear elasticity is introduced to show the general underlying difficulties due to the presence of metamaterials (section 2). We then introduce the Maxwell's equations we are going to study and give two generic well-posedness for these equations that are not equivalent (section 3). The first one (section 3.1) can be applied to study electromagnetic effect into scalar chiral or bi-anisotropic metamaterials. The second one (section 3.2) can handle Maxwell's equation in presence of anisotropic metamaterials. Following the same sketch, we derive generic well-posedness results respectively for acoustics (section 4) and elastic metamaterials (section 5). Finally (section 6), in order to show the interest of these results, we will apply them to study some physical examples of metamaterials.

2. General mathematical framework. We present here in a very general setting the difficulties coming from the presence of metamaterials and the main tools used in the proof of our results.

Let Ω be a simply connected and bounded open set of $\mathbb{R}^N$ with $\mathcal{C}^1$ boundary. The exterior unit normal vector is denoted by $\nu = (\nu_1, \nu_2, \nu_3)$. Let $\mathbb{S} = \sum_{j=1}^{N} \mathcal{S}_j \partial_j$ be a first order differential operator where $\mathcal{S}_j \in Hom(\mathbb{C}^k)$ are linear applications from $\mathbb{C}^k$ to $\mathbb{C}^k$ satisfying $\mathcal{S}_j^* = \mathcal{S}_j$. We now consider the following system:

$$\begin{aligned} &Find\ u \in \mathcal{H}_{\mathbb{S}}\ such\ that: \\ &\left\{ \begin{array}{l} K(p,x)u + \mathbb{S}u(p,x) = f(x),\ x \in \Omega, \\ u(p,x) \in \ker N(x),\ x \in \partial\Omega, \end{array} \right. \end{aligned} \tag{2.1}$$

where $p = iw + \delta$ is the Laplace variable, $N(x)$ is a smooth varying linear application, f is a source term, $K(p,x) \in Hom(\mathbb{C}^k)$, $u(x) \in \mathbb{C}^k$ is an unknown physical quantity (electric or magnetic fields for instance) and $\mathcal{H}_{\mathbb{S}} = \left\{ u \in L^2(\Omega)^k \mid \mathbb{S}u \in L^2(\Omega)^k \right\}$. We note $\mathcal{D}(\mathbb{S})$ for the domain of the unbounded operator $\mathbb{S}$:

$$\mathcal{D}(\mathbb{S}) = \{ u \in \mathcal{H}_{\mathbb{S}} \mid u(x) \in \ker N(x) \text{ for } x \in \partial\Omega \}.$$

The boundary conditions are assumed to verify the conditions given in [31] to ensure maximal dissipativity of the unbounded operator $(\mathbb{S}, \mathcal{D}(\mathbb{S}))$. As a consequence, when the multiplicative operator $K(p,.)$ is (uniformly in the second variable) bounded and coercive, equation (2.1) has an unique solution continuously depending on the source term.

However, when metamaterials are strictly embedded into Ω, the application $K(p,x)$ can no longer be coercive (nor having any specific sign as x travels through Ω) for some p. Hence we cannot deduce existence and uniqueness of solution for (2.1) from the previous argument. This paper is thus devoted to prove well-posedness of systems modelling wave propagation through metamaterials in electromagnetism or optics (theorem 3.2), acoustics (theorem 4.1) and linear elasticity (theorem 5.1) when the tensor $K(p,x)$ verifies the following hypotheses:

Assumptions 1 (General assumptions).
(H1) The application $p \in D_0 \longmapsto K(p,x)$ is holomorphic for almost all $x \in \Omega$, where D_0 is a connected open set of $\mathbb{C}$.
(H2) The application $x \in \overline{\Omega} \longmapsto K(p,x)$ belong to $L^\infty(\overline{\Omega})$ (or replacing by Lipchitz continuous which will be later specified) for all $p \in D_0$ and $K(p,x)^{-1}$ exists for almost all $x \in \overline{\Omega}$.
(H3) There exists p_0 in D_0 such that $K(p_0,.)$ is coercive.

The above assumptions are the basics of our framework to study system (2.1) with metamaterials. Actually $(H1)$ describes the frequency dependence of the physical parameters of the metamaterials. Note that for parameters admitting poles in p, we can restrict D_0 to a subset of the complementary of the union of neighbourhoods of these poles. $(H2)$ is nothing but a physical assumption saying that the homogenized metamaterials have smoothly varying (in $x \in \Omega$) physical parameters. $(H3)$ says that the material behaves like a "physical" one (that is when the parameters are positive definite) for a given $p = p_0$. Furthermore, $(H1)$ and $(H3)$ are motivated by the remark of V.G. Veselago [34] mentioned in the introduction and some examples extracted from the literature [17, 32, 10]. In particular, these hypotheses yield the well-posedness of (2.1) for p_0, see [31].

Let us present now the sketch of the proof of the results presented here. The basic idea behind the method used to bypass the difficulty relying on the "negativity" of $K(p,.)$ is to find a way to extend the existence and uniqueness of solution to (2.1) from p_0 to others p lying in the holomorphy domain of $K(p,.)$. The main tool to do so can be found with the Fredholm's analytical theory ([19] p. 371).

Firstly, for the systems we work with, there exists a first order differential operator $\mathbb{Q}$ such that $\mathbb{Q}\mathbb{S} = 0$ in the sense of distribution and $\ker(\mathbb{S}) = \overline{\mathrm{Im}(\mathbb{Q}^*)}$ where $\ker(\mathbb{S}) = \left\{u \in L^2(\Omega)^k \mid \mathbb{S}u = 0,\ \mathcal{D}'(\Omega)\right\}$. Moreover, they are shown to be subjected to some coercive inequality [24] of the form:

$$\|u\|_{H^1(\Omega)} \leq C\left\{\|u\|_{L^2(\Omega)} + \|\mathbb{S}u\|_{L^2(\Omega)} + \|\mathbb{Q}u\|_{L^2(\Omega)}\right\} \tag{2.2}$$

where $u \in \mathcal{D}(\mathbb{S}) \cap \left\{u \in L^2(\Omega)^k \mid \mathbb{Q}u \in L^2(\Omega)\right\}$. Then, to obtain some compactness for the resolvent of $(\mathbb{S}, \mathcal{D}(\mathbb{S}))$ one has to control $\mathbb{Q}u$ for $u \in \mathcal{D}(\mathbb{S})$. One way to proceed is to use an orthogonal splitting of $L^2(\Omega)^k$ given by a suitable Hodge decomposition [9, 4]:

$$L^2(\Omega)^k = \mathcal{R} \oplus \mathcal{R}^\perp,$$

where $\mathcal{R}$ is a linear subspace of $L^2(\Omega)^k$ such that $\ker(\mathbb{S}) \subset \mathcal{R}$. This implies that $\ker(\mathbb{S})^\perp = \ker(\mathbb{Q}) \supset \mathcal{R}^\perp$. With such a decomposition at hands, one can project (2.1) to get the following equivalent system:

$$\begin{array}{l} \textit{Find } u = P_{\mathcal{R}}u + P_{\mathcal{R}^\perp}u \in \mathcal{D}(\mathbb{S}) \textit{ such that}: \\ \left\{ \begin{array}{l} P_{\mathcal{R}^\perp}K(p,.)(P_{\mathcal{R}}u + P_{\mathcal{R}^\perp}u) + P_{\mathcal{R}^\perp}\mathbb{S}P_{\mathcal{R}^\perp}u = P_{\mathcal{R}^\perp}f, \text{ on } \Omega, \\ P_{\mathcal{R}}K(p,.)(P_{\mathcal{R}}u + P_{\mathcal{R}^\perp}u) = P_{\mathcal{R}}f, \text{ on } \Omega, \end{array} \right. \end{array}$$

where $P_{\mathcal{R}} : L^2(\Omega)^k \longrightarrow \mathcal{R}$ and $P_{\mathcal{R}^\perp} : L^2(\Omega)^k \longrightarrow \mathcal{R}^\perp$ are the orthogonal projections associated to the Hodge decomposition. Remark then that one needs to invert the family of operators $P_{\mathcal{R}^\perp}K(p,.)P_{\mathcal{R}^\perp} + P_{\mathcal{R}^\perp}\mathbb{M}P_{\mathcal{R}^\perp}$ on $\mathcal{R}^\perp \cap \mathcal{D}(\mathbb{S})$. As this family is holomorphic (thanks to (H1)) and have compact resolvent (thanks to inequality 2.2), this inversion can be done with help of Fredholm analytic theory. This yields to existence and uniqueness of $P_{\mathcal{R}^\perp}u(p,.)$ for all $p \in D_0\backslash S$ for S a set of exceptional values. However, what is difficult is to get $P_{\mathcal{R}}u$ in this very general setting. Indeed, one has to invert the operator $P_{\mathcal{R}}K(p,.)P_{\mathcal{R}}$ on $\mathcal{R}$ for all $p \in D_0$, even when $K(p,.)$ is not coercive. To do this, one must have more information on the space $\mathcal{R}$ implying that we must specify the physics we work with. We thus prove theorems 3.2, 4.1 and 5.1 but under restrictive phenomenological assumptions (scalar valued physical parameters for instance). In some others cases we can prove the invertibilty of $P_{\mathcal{R}}K(p,.)P_{\mathcal{R}}$ (leading to theorems 3.6, 4.5 and 5.2) by adding one hypothesis of the form

Assumptions 2 (General additional assumptions for non-scalar parameters).
(H4) There exists $a(p,x) \in \mathbb{C}$, Lipschitz continuous for $x \in \overline{\Omega}$ and holomorphic for $p \in D_0$, such that $K(p,x)a(p,x)$ is coercive for all $p \in D_0$ and almost all $x \in \Omega$.

Remark 2.1. *Assumption $(H4)$ is added when tensorial and not scalar parameters are considered because of change of the spectral properties of the operator involved. For instance, Maxwell's equations in chiral media may admit an essential spectrum close to zero on the positive imaginary axis. This does not occur in isotropic media [22]. Finally note that assumption $(H4)$ is satisfied for scalar Lipchitz continuous parameters satisfying $(H1)-(H2)$. Indeed, in that case, $(H4)$ is satisfied with $a(p,x) = K(p,x)^{-1}$.*

3. Generic well-posedness results for electromagnetism and optics. In this section, we study electromagnetic phenomenons in presence of metamaterials with three different models. Firstly we look at the usual time-harmonic Maxwell's equations in Laplace transform:

$$\begin{array}{l} \textit{Find } (e,h) \in H(\mathrm{curl},\Omega)^2 \textit{ such that}: \\ \left\{ \begin{array}{l} p\varepsilon(p,x)e(p,x) - \nabla \times h(p,x) = -j(x), \text{ on } \Omega, \\ p\mu(p,x)h(p,x) + \nabla \times e(p,x) = -m(x), \text{ on } \Omega, \\ \nu(x) \times (e(p,x) + \Lambda(x)(\nu(x) \times h(p,x))) = 0, \ x \in \partial\Omega, \end{array} \right. \end{array} \tag{3.1}$$

where $H(\mathrm{curl},\Omega) = \left\{\Psi \in L^2(\Omega)^3 \mid \nabla \times \Psi \in L^2(\Omega)^3\right\}$, e is the electric field, h is the magnetic field, ε stands for the permittivity, μ for the permeability and j and m are respectively the electric and magnetic current densities. At last $\Lambda : \partial\Omega \longrightarrow Hom(\mathbb{C}^3)$ describes impedance boundary condition and is assumed to be Lipschitz continuous and coercive:

$$\forall\, z \in \mathbb{C}^3, \mathcal{R}e \left\langle (\Lambda + \Lambda^*)\, z, \overline{z} \right\rangle \geq 0,$$

noting $\langle X, Y \rangle = \sum_{j=1}^N X_j Y_j$ the standard scalar product of vector $X, Y \in \mathbb{C}^N$ with associated norm $|X| = \sqrt{\langle X, X \rangle}$. Note that the tangential traces on $\partial\Omega$ of $(e,h) \in H(\mathrm{curl},\Omega)^2$ have to be understood in the sense that $\left(\nu \times e_{|\partial\Omega}, \nu \times h_{|\partial\Omega}\right) \in H^{-\frac{1}{2}}(\partial\Omega)^2$.

A second model we are interested in describes the wave propagation of time-harmonic electromagnetic waves through bi-anisotropic media [21]:

$$\begin{array}{l} \textit{Find } (e,h) \in H(\mathrm{curl},\Omega)^2 \textit{ such that}: \\ \left\{ \begin{array}{l} p\varepsilon(p,x)e(p,x) + p\xi(p,x)h(p,x) - \nabla \times h(p,x) = -j(x), \text{ on } \Omega, \\ p\mu(p,x)h(p,x) + p\zeta(p,x)e(p,x) + \nabla \times e(p,x) = -m(x), \text{ on } \Omega, \\ \nu(x) \times (e(p,x) + \Lambda(x)(\nu(x) \times h(p,x))) = 0, \ x \in \partial\Omega, \end{array} \right. \end{array} \tag{3.2}$$

where ξ and ζ are called the coupling constants.

Finally, in the special case of chiral media appearing in optics for crystals, one can replace Maxwell's equations by the Drude-Born-Fedorov system [17]:

$$\text{Find } (e,h) \in H(\mathrm{curl},\Omega)^2 \text{ such that :}\\ \begin{cases} p\varepsilon(p,x)e(p,x) + p\beta(p,x)\varepsilon(p,x)\nabla\times e(p,x) \\ \qquad\qquad -\nabla\times h(p,x) = -j(x),\ x\in\Omega, \\ p\mu(p,x)h(p,x) + p\beta(p,x)\mu(p,x)\nabla\times h(p,x) \\ \qquad\qquad +\nabla\times e(p,x) = -m(x),\ x\in\Omega, \\ \nu(x)\times(e(p,x)+\Lambda(x)(\nu(x)\times h(p,x))) = 0,\ x\in\partial\Omega, \end{cases} \tag{3.3}$$

where β is the chirality of the material embedded into Ω.

Remark 3.1. *Maxwell's equations (3.1) are special case of (3.2) and (3.3) that can be obtained respectively by taking $\beta = 0$ or $\xi = \zeta = 0$.*

Moreover, (3.2) and (3.3) are equivalent up to some algebraic computations. Hence, we will give two sets of assumptions on the media embedded into Ω according whether it is scalar bi-anisotropic (assumption 3) or chiral (assumption 4) but only one theorem will be formulated (theorem 3.2). To show that (3.3) is similar to (3.2) (the reciprocal can be done in the same way), first rewrite (3.3) as follows:

$$\begin{cases} \nabla\times h = p\varepsilon e + p\beta\varepsilon\nabla\times e + j, \\ \nabla\times e = -m - p\mu h - p\beta\mu\nabla\times h, \end{cases}$$

where the dependence of the physical parameters in (p,x) have been dropped to lighten the overall expressions. Then, reporting the first equation above into the second one of (3.3) and vise-versa, one gets:

$$\begin{cases} p\varepsilon e + p\beta\varepsilon\left(-m - p\mu h - p\beta\mu\nabla\times h\right) - \nabla\times h = -j, \\ p\mu h + p\beta\mu\left(p\varepsilon e + p\beta\varepsilon\nabla\times e + j\right) + \nabla\times e = -m. \end{cases}$$

Finally, gathering the previous computations, one has the following (3.2) system:

$$\begin{cases} \left(\mathbb{I}_3 + p^2\beta^2\varepsilon\mu\right)^{-1}\left(p\varepsilon e - p^2\beta\varepsilon\mu h\right) - \nabla\times h = -\widetilde{j}, \\ \left(\mathbb{I}_3 + p^2\beta^2\mu\varepsilon\right)^{-1}\left(p\mu h + p^2\beta\mu\varepsilon e\right) + \nabla\times e = -\widetilde{m}. \end{cases}$$

We are going to give two generic well-posedness results (that is existence and uniqueness of solution except for some p) for electromagnetism. The first generic well-posedness result allows to study either chiral (3.3) or bi-anisotropic (3.2) materials having scalar physical parameters satisfying assumptions similar to $(H1)-(H2)-(H3)$. The second result, for 3×3 tensorial coefficients, requires additionally a variant of assumption $(H4)$.

3.1. Generic well-posedness for scalar chiral or bi-anisotropic materials. We focus here on solving equation (3.3) or (3.2) in presence of materials characterized by scalar parameters, and introduce the following assumptions:

Assumptions 3 (For scalar bi-anisotropic materials).

(B1) The applications $\varepsilon(p,x), \mu(p,x), \xi(p,x)$ and $\zeta(p,x)$ are holomorphic in $p \in D_0$ for almost all $x \in \Omega$, where D_0 is a connected open set of $\mathbb{C}$.

(B2) *The applications $\varepsilon(p,x), \mu(p,x), \xi(p,x)$ and $\zeta(p,x)$ are Lipschitz continuous in $x \in \overline{\Omega}$ for all $p \in D_0$. Moreover, $\varepsilon(p,x)\mu(p,x) - \xi(p,x)\zeta(p,x) \neq 0$ for almost all $x \in \overline{\Omega}$ and for any $p \in D_0$.*

(B3) *There exists p_0 in D_0 and $\alpha > 0$ such that the following inequality holds:*

$$\begin{aligned}&\mathcal{R}e\left(< p_0\varepsilon(p_0,x)X, \overline{X} > + < p_0\mu(p_0,x)Y, \overline{Y} > + < p_0\xi(p_0,x)Y, \overline{X} >\right)\\ &+\mathcal{R}e\left(< p_0\zeta(p_0,x)X, \overline{Y} >\right) \geq \alpha(|X|^2 + |Y|^2),\end{aligned}$$

for almost all $x \in \Omega$ and for all $X, Y \in \mathbb{C}^3$.

Assumptions 4 (For scalar chiral media).

(C1) *The applications $\varepsilon(p,x), \mu(p,x)$ and $\beta(p,x)$ are holomorphic in $p \in D_0$ for almost all $x \in \Omega$, where D_0 is a connected open set of $\mathbb{C}$.*

(C2) *The applications $\varepsilon(p,x), \mu(p,x)$ and $\beta(p,x)$ are Lipschitz continuous in $x \in \overline{\Omega}$ for all $p \in D_0$. Moreover, $p^2\varepsilon(p,x)\mu(p,x)M(p,x) \neq 0$, with $M(p,x) = (1 + p^2\beta(p,x)^2\varepsilon(p,x)\mu(p,x))^{-1}$, for almost all $x \in \overline{\Omega}$.*

(C3) *There exist p_0 in D_0 and $\alpha > 0$ such that, for almost all $x \in \Omega$ and $X, Y \in \mathbb{C}^3$:*

$$\begin{aligned}&\mathcal{R}e\left\{< p_0\varepsilon(p_0,x)M(p_0,x)X, \overline{X} > + < p_0\mu(p_0,x)M(p_0,x)Y, \overline{Y} >\right\}\\ &+\mathcal{R}e\left\{< p_0^2\beta(p_0,x)\varepsilon(p_0,x)\mu(p_0,x)M(p_0,x)Y, \overline{X} >\right\}\\ &-\mathcal{R}e\left\{< p_0^2\beta(p_0,x)\varepsilon(p_0,x)\mu(p_0,x)M(p_0,x)X, \overline{Y} >\right\} \geq \alpha(|X|^2 + |Y|^2),\end{aligned}$$

Finally, let us note $H(\mathrm{div},\Omega) = \left\{e \in L^2(\Omega)^3 \mid \mathrm{div} e \in L^2(\Omega)\right\}$. Then, the first main result of this paper is:

Theorem 3.2. *Suppose that $(j,m) \in (H(\mathrm{div},\Omega))^2$ and that assumption 3 (respectively 4) are satisfied. Then Maxwell's equations (3.2) (respectively (3.3)) are well-posed for all $p \in D_0\backslash S$ where S is a discrete, locally finite and possibly empty set of D_0. Moreover the solution $(e(p,.), h(p,.))$ verifies:*

$$\|(e(p,.),h(p,.))\|_{L^2(\Omega)^6} \leq C(p)\left\{\left\|\begin{pmatrix} j \\ m \end{pmatrix}\right\|_{L^2(\Omega)^6} + \left\|\begin{pmatrix} \mathrm{div}(j) \\ \mathrm{div}(m) \end{pmatrix}\right\|_{L^2(\Omega)^2}\right\},$$

with $C(p)$ a constant depending only on p and Ω. Moreover, the application $p \in D_0\backslash S \longmapsto (e(p,.),h(p,.)) \in L^2(\Omega)^6$ is holomorphic.

Proof. The proof follows the sketch presented in section 2. Hence we start by formulating Maxwell's equations as:

$$\begin{aligned}&Find\ u = (e,h)^T \in \mathcal{H}_{\mathbb{M}}\ such\ that:\\ &\left\{\begin{array}{l} K(p,x)u(p,x) + \mathbb{M}u(p,x) = f(x),\ x \in \Omega, \\ \nu(x) \times (e(p,x) + \Lambda(x)(\nu(x) \times h(p,x))) = 0,\ x \in \partial\Omega, \end{array}\right.\end{aligned} \tag{3.4}$$

where $f = (-j,-m) \in H(\mathrm{div},\Omega)^2$ and $\mathbb{M}$ is the unbounded operator:

$$\mathbb{M} = \begin{pmatrix} 0 & -\nabla\times \\ \nabla\times & 0 \end{pmatrix},$$

with domain $\mathcal{D}(\mathbb{M})$ defined by:

$$\mathcal{D}(\mathbb{M}) = \left\{(e,h)^T \in (H(\mathrm{curl},\Omega))^2 \mid \nu(x) \times \left(e_{|\partial\Omega} + \Lambda(x)(\nu(x) \times h_{|\partial\Omega})\right) = 0,\ x \in \partial\Omega\right\}.$$

Note that $(\mathbb{M}, \mathcal{D}(\mathbb{M}))$ is maximal dissipative. Finally, $K(p,x) \in Hom(\mathbb{C}^6)$ is given by:

$$K(p,x) = \begin{pmatrix} K_{11}(p,x)\mathbb{I}_3 & K_{12}(p,x)\mathbb{I}_3 \\ K_{21}(p,x)\mathbb{I}_3 & K_{22}(p,x)\mathbb{I}_3 \end{pmatrix}, \tag{3.5}$$

where $\mathbb{I}_N$ denotes the identity operator of $\mathbb{C}^N$, and with

- for bi-anisotropic media (3.2): $K_{11}(p,x) = p\varepsilon(p,x)$, $K_{12}(p,x) = p\xi(p,x)$, $K_{21}(p,x) = p\zeta(p,x)$, $K_{22}(p,x) = p\mu(p,x)$,
- and for chiral media (3.3): $K_{11}(p,x) = p\varepsilon(p,x)M(p,x)$, $K_{21}(p,x) = -K_{12}(p,x)$, $K_{12}(p,x) = p^2\beta(p,x)\varepsilon(p,x)\mu(p,x)M(p,x)$, $K_{22}(p,x) = p\mu(p,x)M(p,x)$ where $M(p,x)$ is defined in $(C3)$.

Note $\mathbb{Q} = \begin{pmatrix} \text{div} & 0 \\ 0 & \text{div} \end{pmatrix}$. Then, using that $\mathbb{Q}\mathbb{M} = 0$ and that $K(p,.)$ satisfies $(B2)$ allows to take the divergence of equation (3.4) to obtain:

$$\mathbb{Q}f = \mathbb{Q}K(p,x)u + \mathbb{Q}\mathbb{M}u \quad =: Z(p,x)u + \widetilde{K}(p,x)\mathbb{Q}u.$$

In this last identity, $\widetilde{K}(p,x) \in Hom(\mathbb{C}^2)$ and $Z(p,x) \in Hom(\mathbb{C}^6, \mathbb{C}^2)$ are given by:

$$\widetilde{K}(p,x) = \begin{pmatrix} K_{11}(p,x) & K_{12}(p,x) \\ K_{21}(p,x) & K_{22}(p,x) \end{pmatrix},$$
$$Z(p)u = Z(p)\begin{pmatrix} e \\ h \end{pmatrix} = \begin{pmatrix} \langle \nabla K_{11}(p,x), e\rangle + \langle \nabla K_{12}(p,x), h\rangle \\ \langle \nabla K_{21}(p,x), e\rangle + \langle \nabla K_{22}(p,x), h\rangle \end{pmatrix}.$$

An elliptization of Maxwell's equations is then performed by writing a relaxed version of system (3.4) by the introduction of two new unknown functions φ and ψ - identically zero in (3.4)- as follows [22]:

$$\text{Find } (e,h,\varphi,\psi) \in \mathcal{D}(\mathbb{M}) \cap (H(\text{div},\Omega))^2 \times H_0^1(\Omega)^2 \text{ such that :}$$
$$\begin{cases} (K(p,x) + \mathbb{M})\begin{pmatrix} e \\ h \end{pmatrix} + \begin{pmatrix} \nabla\varphi \\ \nabla\psi \end{pmatrix} = f, \ x \in \Omega, \\ \widetilde{K}(p,x)^{-1}Z(p,x)\begin{pmatrix} e \\ h \end{pmatrix} + \mathbb{Q}\begin{pmatrix} e \\ h \end{pmatrix} \\ \qquad\qquad + \widetilde{K}(p,x)^{-1}\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \widetilde{K}(p,x)^{-1}\mathbb{Q}f. \end{cases} \tag{3.6}$$

As $\det(K(p,x)) = \det(\widetilde{K}(p,x))^3$, $\widetilde{K}(p,x)$ is invertible thanks to $(B2)$ and the above equation makes sense. Now introduce the closed unbounded operator $\mathbb{T}$:

$$\mathbb{T} = \begin{pmatrix} 0 & -\nabla\times & \nabla & 0 \\ \nabla\times & 0 & 0 & \nabla \\ \text{div} & 0 & 0 & 0 \\ 0 & \text{div} & 0 & 0 \end{pmatrix},$$

with domain:

$$\mathcal{D}(\mathbb{T}) = \left\{ U \in (H(\text{curl},\Omega) \cap H(\text{div},\Omega))^2 \times H^1(\Omega)^2 \mid U(x)_{|\partial\Omega} \in \ker(\widetilde{N}(x)) \right\},$$

for $U = (e,h,\varphi,\psi)$ and where the boundary conditions are encoded by the linear application $\widetilde{N} : \partial\Omega \longrightarrow Hom(\mathbb{C}^8, \mathbb{C}^5))$ which is Lipschitz continuous and defined by:

$$\widetilde{N}(x)U(x) = \begin{pmatrix} \nu \times e(x) + \nu \times \Lambda(x)(\nu \times h(x)) \\ \varphi(x) \\ \psi(x) \end{pmatrix}, \text{ for } x \in \partial\Omega.$$

The elliptized Maxwell's equations (3.6) can now be summarized as follows:

$$\text{Find } U=(e,h,\varphi,\psi)\in\mathcal{H}_{\mathbb{T}} \text{ such that :}\quad \begin{cases} (\widetilde{K}(p,x)+\mathbb{T})U(p,x)=F(p,x),\ x\in\Omega,\\ U(p,x)\in\ker(\widetilde{N}(x)),\ x\in\partial\Omega, \end{cases} \tag{3.7}$$

where $\widetilde{K}(p,.)$ belongs to $L^\infty(\overline{\Omega},Hom(\mathbb{C}^8))$ according to $(B1)$ and Rademacher's theorem. Furthermore $F(p,.):=\left(f,\widetilde{K}(p,.)^{-1}\mathbb{Q}f\right)\in L^2(\Omega)^8$ is holomorphic for all $p\in D_0$.

Lemma 3.3. *Assume that* $F(p,.)=\left(f,\widetilde{K}(p,.)^{-1}\mathbb{Q}f\right)$ *for* $f\in(H(\mathrm{div},\Omega))^2$ *and* $U=(e,h,\varphi,\psi)\in\mathcal{D}(\mathbb{T})$ *satisfies (3.7). Then* $\varphi=\psi=0$ *and* $u=(e,h)\in\mathcal{D}(\mathbb{M})$ *is solution to (3.4).*

From lemma 3.3, (3.4) corresponds to solving (3.7). Moreover, we have:

Lemma 3.4. $(\mathbb{T},\mathcal{D}(\mathbb{T}))$ *is maximal dissipative with compact resolvent.*

From lemma 3.4 and as the multiplicative operator $\widetilde{K}(p,.)$ is bounded for all p in D_0, the resolvent set of $\left(\widetilde{K}(p,.)+\mathbb{T},\mathcal{D}(\mathbb{T})\right)$ is thus non-empty for all $p\in D_0$. Moreover, the holomorphic family of closed operators $\left(\widetilde{K}(p,.)+\mathbb{T},\mathcal{D}(\mathbb{T})\right)_{p\in D_0}$ has compact resolvent. Hence, solving (3.6) is the same as inverting a holomorphic family of closed operators with compact resolvent. This can actually be done with Fredholm analytical theory ([19] theorem 1.10 p. 371) since we find $p_0\in D_0$ such that (3.6) is well-posed. This is the purpose of the lemma below.

Lemma 3.5. *The operator* $\left(\widetilde{K}(p_0,.)+\mathbb{T},\mathcal{D}(\mathbb{T})\right)$, *for* p_0 *satisfying* $(B3)$ *(respectively* $(C3)$*), is invertible and* $\left(\widetilde{K}(p_0,.)+\mathbb{T}\right)^{-1}\in\mathcal{B}\left(L^2(\Omega)^8,\mathcal{D}(\mathbb{T})\right)$.

The notation $\mathcal{B}(\mathbf{X},\mathbf{Y})$ stands for the set of bounded linear operator from $\mathbf{X}$ to $\mathbf{Y}$, both being Banach spaces. Using lemma 3.5 and Fredholm analytical theory, we obtain that the holomorphic family of closed operators with compact resolvent $\left(\widetilde{K}(p,.)+\mathbb{T},\mathcal{D}(\mathbb{T})\right)_{p\in D_0}$ is invertible for $p\in D_0\backslash S$ where S is a discrete, locally finite and possibly empty set of D_0. This shows that the system (3.7) is well-posed for $p\in D_0\backslash S$. Furthermore, using $(B1)$ and $(B2)$ (respectively $(C1)-(C2)$), the applications $p\in D_0\longmapsto\left(\widetilde{K}(p,.)+\mathbb{T}\right)\in\mathcal{B}(\mathcal{D}(\mathbb{T}),L^2(\Omega)^8)$ and $p\in D_0\longmapsto F(p)\in L^2(\Omega)^8$ are both holomorphic and so is (see [19] p.365) the application: $U:p\in D_0\backslash S\longmapsto U(p,.):=\left(\widetilde{K}(p,.)+\mathbb{T}\right)^{-1}F(p,.)\in L^2(\Omega)^8$. Consequently the application solution to (3.7) $p\in D_0\backslash S\longmapsto u(p,.)\in L^2(\Omega)^6$, where $u(p,.)$ satisfies (3.4) is holomorphic. Moreover we get, for some constant $C(p)>0$, the estimate:

$$\left\|U(p,.)\right\|_{L^2(\Omega)^8}\le C(p)\left\|F(p,.)\right\|_{L^2(\Omega)}.$$

Now using lemma 3.3 we obtain that the previous estimate reduces to:

$$\left\|e(p,.)\right\|_{L^2(\Omega)^3}+\left\|h(p,.)\right\|_{L^2(\Omega)^3}\le C(p)\left\{\left\|f\right\|_{L^2(\Omega)}+\left\|\mathbb{Q}f\right\|_{L^2(\Omega)}\right\},$$

which conclude the proof. □

The rest of the section is dedicated to the proof of the lemmas set up above.

Proof. [**Proof of lemma 3.3**] We follow the proof of theorem 4.2 in [22]. Applying $\mathbb{Q}$ to the first equation of (3.6) shows that:

$$\begin{pmatrix} \Delta\varphi \\ \Delta\psi \end{pmatrix} = \mathbb{Q}f - \mathbb{Q}\left(K(p,.)\begin{pmatrix} e \\ h \end{pmatrix}\right),$$

where $\Delta\psi = \text{div}\,(\nabla\psi)$. From the definition of $\widetilde{K}(p,x)$ and $Z(p,x)$, we derive that the second equation of (3.6) can be written as follows:

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \mathbb{Q}f - \mathbb{Q}\left(K(p,.)\begin{pmatrix} e \\ h \end{pmatrix}\right).$$

As a result, φ and ψ are both solutions to the following equation:

$$\begin{array}{l} \textit{Find } \vartheta \in H_0^1(\Omega) \textit{ such that } : \\ \quad \Delta\vartheta - \vartheta = 0, \text{ on } \Omega, \end{array}$$

for $\vartheta \in \{\varphi, \psi\}$, which only admits the null solution. □

Proof. [**Proof of lemma 3.4**] There exists three symmetric tensors $T_j \in Hom(\mathbb{C}^8)$ such that $\mathbb{T} = \sum_{j=1}^{3} T_j\partial_j$. Thus, according to [31], the unbounded operator $(\mathbb{T}, \mathcal{D}(\mathbb{T}))$ is maximal dissipative if:

(a) On every connected component of $\partial\Omega$, the rank of $\mathbb{T}_\nu(x) := \sum_{j=1}^{3} T_j\nu_j(x)$, $x \in \partial\Omega$ is constant,

(b) For all $x \in \partial\Omega$ and $U(p,x) \in \ker(\widetilde{N}(x))$, we have $\left\langle \mathbb{T}_\nu(x)U(p,x), \overline{U(p,x)} \right\rangle \geq 0$,

(c) $\dim(\ker(\widetilde{N}(x)) = \sharp\{\text{non} - \text{negative eigenvalues of } \mathbb{T}_\nu \text{ counting multiplicity}\}$.

Firstly, straightforward computations shows that $\det(\mathbb{T}_\nu) = \left(\nu_1^2 + \nu_2^2 + \nu_3^2\right)^4 > 0$ and then (a) is satisfied. Then, for $U(x) = (e, h, \varphi, \psi) \in \ker(\widetilde{N}(x))$, it comes:

$$\left\langle \mathbb{T}_\nu(x)U(x), \overline{U(x)} \right\rangle = 2\mathcal{R}e\left\langle \Lambda(x)(\nu \times h), \overline{\nu \times h} \right\rangle \geq 2\alpha|\nu \times h|^2 \geq 0,$$

verifying (b). Finally the spectrum of $\mathbb{T}_\nu$ is $\sigma(\mathbb{T}_\nu) = \{-1, +1\}$ both with multiplicity 4 and $\dim(\ker(\widetilde{N}(x))) = 4$, so (c) is satisfied too.

Now, from [24], the following inequality holds for all $u \in \mathcal{D}(\mathbb{M}) \cap (H(\text{div}, \Omega))^2$

$$\|u\|_{(H^1(\Omega))^6} \leq C\left\{\|u\|_{(L^2(\Omega))^6} + \|\mathbb{M}u\|_{(L^2(\Omega))^6} + \|\mathbb{Q}u\|_{(L^2(\Omega))^6}\right\}.$$

Thus, the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ being compact, one obtains that the embedding of $\mathcal{D}(\mathbb{T})$ into $L^2(\Omega)^8$ is compact. Hence, the resolvent of $(\mathbb{T}, \mathcal{D}(\mathbb{T}))$ is compact. □

Proof. [**Proof of lemma 3.5**] Inverting $\left(\widetilde{K}(p_0,.) + \mathbb{T}\right)$ on $\mathcal{D}(\mathbb{T})$ means solving the following problem:

$$\begin{array}{l} \textit{Find } U = (e, h, \varphi, \psi) \in \mathcal{H}_{\mathbb{T}} \textit{ such that} : \\ \begin{cases} (\widetilde{K}(p_0, x) + \mathbb{T})U(p, x) = G, \ x \in \Omega, \\ U(p, x) \in \ker(\widetilde{N}(x)), \ x \in \partial\Omega, \end{cases} \end{array} \tag{3.8}$$

for G in $L^2(\Omega)^8$. From lemma 3.4, there exists at least one $\alpha > 0$ belonging to the resolvent set of the operator $(\mathbb{T}, \mathcal{D}(\mathbb{T}))$. Furthermore the resolvent operator $(\alpha\mathbb{I}_8 + \mathbb{T})^{-1}$ is a compact operator of $L^2(\Omega)^8$. Then (3.8) remains to solve the following problem:

$$\begin{array}{l} \textit{Find } U \in L^2(\Omega)^8 \textit{ such that} \\ \left(\mathbb{I}_8 + (\alpha\mathbb{I}_8 + \mathbb{T})^{-1}\left(\widetilde{K}(p_0,.) - \alpha\mathbb{I}_8\right)\right) U = (\alpha\mathbb{I}_8 + \mathbb{T})^{-1} G \quad =: \widetilde{G}. \end{array} \tag{3.9}$$

The boundedness of $\widetilde{K}(p_0, x)$ shows that the operator $(\alpha\mathbb{I}_8 + \mathbb{T})^{-1}\left(\widetilde{K}(p_0,.) - \alpha\mathbb{I}_8\right)$ is compact on $L^2(\Omega)^8$. Then, from the Fredholm alternative, we only need to show the injectivity of the operator $\left(\mathbb{I}_8 + (\alpha\mathbb{I}_8 + \mathbb{T})^{-1}\left(\widetilde{K}(p_0,.) - \alpha\mathbb{I}_8\right)\right)$ acting on $L^2(\Omega)^8$ to prove the lemma. Taking $\widetilde{G} = 0$ in (3.9) and using the boundedness of $(\alpha\mathbb{I}_8 + \mathbb{T})^{-1} : L^2(\Omega)^8 \longmapsto \mathcal{D}(\mathbb{T})$ implies that U belongs to $\mathcal{D}(\mathbb{T})$ and satisfies (3.8) with right member $G = 0$. Now lemma 3.3 shows that $U = (e, h, \varphi, \psi) = (e, h, 0, 0)$ with (e, h) verifying (3.4) for $p = p_0$. Then from $(B3)$ (respectively $(C3)$) and the uniqueness of solution to (3.4) we infer that $(e, h) = (0, 0)$ concluding the proof of the lemma. □

3.2. Generic well-posedness for some anisotropic materials. The generic well-posedness result for the Maxwell systems (3.1), (3.3) or (3.2) proved in the previous section can be only applied to materials with scalar physical parameters. In the case of anisotropic materials, we would have to take into account for some physical parameters that are no longer scalar. This done in this section. We start introducing the corresponding assumptions:

Assumptions 5 (For bi-anisotropic materials having tensorial parameters).

(BT1) *The applications $\varepsilon(p,x), \mu(p,x), \xi(p,x)$ and $\zeta(p,x)$ (which are now 3×3 tensors) are holomorphic in $p \in D_0$ for almost all $x \in \Omega$, where D_0 is a connected open set of $\mathbb{C}$.*

(BT2) *The applications $\varepsilon(p,x), \mu(p,x), \xi(p,x)$ and $\zeta(p,x)$ belong to $L^\infty(\overline{\Omega})$ for all $p \in D_0$. Moreover, $\det(\varepsilon(p,x)\mu(p,x) - \xi(p,x)\zeta(p,x)) \neq 0$ for almost all $x \in \overline{\Omega}$ and for any $p \in D_0$.*

(BT3) *There exists $p_0 \in D_0$ and $\alpha > 0$ such that the following inequality holds:*

$$\begin{array}{l} \mathcal{R}e\left\{< p_0\varepsilon(p_0,x)X, \overline{X} > + < p_0\mu(p_0,x)Y, \overline{Y} > + < p_0\xi(p_0,x)Y, \overline{X} >\right\} \\ +\mathcal{R}e\left\{< p_0\zeta(p_0,x)X, \overline{Y} >\right\} \geq \alpha(|X|^2 + |Y|^2), \end{array}$$

for almost all $x \in \Omega$ and for all $X, Y \in \mathbb{C}^3$.

(BT4) *There exists $a(p,x) \in \mathbb{C}$, Lipschitz continuous for $x \in \overline{\Omega}$ and holomorphic for $p \in D_0$ verifying:*

$$\begin{array}{l} \mathcal{R}e\left\{< p\varepsilon(p,x)a(p,x)X, \overline{X} > + < p\mu(p,x)a(p,x)Y, \overline{Y} >\right\} \\ +\mathcal{R}e\left\{< p\xi(p,x)a(p,x)Y, \overline{X} > + < p\zeta(p,x)a(p,x)X, \overline{Y} >\right\} \geq \alpha(|X|^2 + |Y|^2), \end{array}$$

for all $p \in D_0$, for almost all $x \in \Omega$ and for all $X, Y \in \mathbb{C}^3$.

Assumptions 6 (For dielectric materials having tensorial parameters). *Suppose that (BT1)-(BT2)-(BT3) are holding and replace (BT4) with:*

(DT4) *Parameters ξ and ζ are vanishing, and there exists $a_\varepsilon(p,x),\ a_\mu(p,x) \in \mathbb{C}$ both Lipschitz continuous for $x \in \overline{\Omega}$ and holomorphic for $p \in D_0$. Moreover, there exists $\alpha > 0$ such that following inequalities holds:*

$$\mathcal{R}e\left\{< p\vartheta(p,x)a_\vartheta(p,x)X, \overline{X} >\right\} \geq \alpha|X|^2,$$

for $\vartheta \in \{\varepsilon, \mu\}$, for all $p \in D_0$, for almost all $x \in \Omega$ and for all $X, Y \in \mathbb{C}^3$.

Assumptions 7 (For chiral materials having tensorial parameters).

(CT1) The applications $\varepsilon(p,x), \mu(p,x)$ and $\beta(p,x)$ (which are now 3×3 tensors) are holomorphic in $p \in D_0$ for almost all $x \in \Omega$, where D_0 is a connected open set of $\mathbb{C}$.

(CT2) The applications $\varepsilon(p,x), \mu(p,x)$ and $\beta(p,x)$ are in $L^\infty(\overline{\Omega})$ for all $p \in D_0$. Moreover, we ask for

$$\det\left[p^2\varepsilon(p,x)\mu(p,x)\right]\det\widetilde{\mathcal{M}} \neq 0,$$

for almost all $x \in \overline{\Omega}$ with $\widetilde{\mathcal{M}}(p,x) = \left(\mathbb{I}_3 + p^2\beta(p,x)\varepsilon(p,x)\beta(p,x)\mu(p,x)\right)^{-1}$.

(CT3) There exist p_0 in D_0 and $\alpha > 0$ such that the following inequality

$$\begin{aligned}
&\mathcal{R}e\left\{< p_0\widetilde{\mathcal{M}}(p_0,x)\varepsilon(p_0,x)X, \overline{X} > + < p_0\mathcal{M}(p_0,x)\mu(p_0,x)Y, \overline{Y} >\right\}\\
&+\mathcal{R}e\left\{< p_0^2\beta(p_0,x)\varepsilon(p_0,x)\mathcal{M}(p_0,x)\mu(p_0,x)Y, \overline{X} >\right\}\\
&-\mathcal{R}e\left\{< p_0^2\widetilde{\mathcal{M}}(p_0,x)\beta(p_0,x)\mu(p_0,x)\varepsilon(p_0,x)X, \overline{Y} >\right\} \geq \alpha(|X|^2 + |Y|^2),
\end{aligned}$$

where $\mathcal{M}(p,x) = \left(\mathbb{I}_3 + p^2\beta(p,x)\mu(p,x)\beta(p,x)\varepsilon(p,x)\right)^{-1}$, holds for almost all $x \in \Omega$ and for all $X, Y \in \mathbb{C}^3$.

(CT4) There exists $a(p,x) \in \mathbb{C}$, Lipschitz continuous for $x \in \overline{\Omega}$ and holomorphic for $p \in D_0$ such that the following inequality holds:

$$\begin{aligned}
&\mathcal{R}e\left\{< pa(p,x)\widetilde{\mathcal{M}}(p,x)\varepsilon(p,x)X, \overline{X} > + < pa(p,x)\mathcal{M}(p,x)\mu(p,x)Y, \overline{Y} >\right\}\\
&+\mathcal{R}e\left\{< p^2a(p,x)\beta(p,x)\varepsilon(p,x)\mathcal{M}(p,x)\mu(p,x)Y, \overline{X} >\right\}\\
&-\mathcal{R}e\left\{< p^2a(p,x)\widetilde{\mathcal{M}}(p,x)\beta(p,x)\mu(p,x)\varepsilon(p,x)X, \overline{Y} >\right\} \geq \alpha(|X|^2 + |Y|^2),
\end{aligned}$$

for all $p \in D_0$, for almost all $x \in \Omega$ and for all $X, Y \in \mathbb{C}^3$.

We can now formulate the following theorem:

Theorem 3.6. *If assumption 5 (respectively 6 or 7) is satisfied, then for all $(j,m) \in L^2(\Omega)^3 \times L^2(\Omega)^3$ Maxwell's system (3.2) (resp. (3.1) or (3.3)) has an unique solution for all p in $D_0\backslash S$ where $S \subset D_0$ is a discrete, locally finite and possibly empty set of D_0. Moreover the solution satisfies the bound:*

$$\forall p \in D_0\backslash S, \|(e(p,.), h(p,.))\|_{L^2(\Omega)} \leq C(p)\,\|(j,m)\|_{L^2(\Omega)},$$

and the application $p \in D_0\backslash S \longmapsto (e(p,.), h(p,.)) \in L^2(\Omega)^6$ is holomorphic.

Proof. We focus on system (3.4) when the multiplicative operator $K(p,x)$ satisfying $(BT1) - (BT2) - (BT3)$ and either $(BT4)$ or $(DT4)$. The case of media checking $(CT1) - (CT2) - (CT3) - (CT4)$ is very similar.

Let us introduce the following Hodge decomposition [9]:

$$L^2(\Omega)^3 \times L^2(\Omega)^3 = (H(\mathrm{div}0,\Omega))^2 \oplus \mathrm{grad}(H_0^1(\Omega))^2, \tag{3.10}$$

where $H(\mathrm{div}0,\Omega) = \left\{v \in L^2(\Omega)^3 \mid \mathrm{div}v = 0\right\}$. and denote by $P_0 : L^2(\Omega)^6 \longrightarrow H(\mathrm{div}0,\Omega)^2$ and $P_\nabla : L^2(\Omega)^6 \longrightarrow \mathrm{grad}(H_0^1(\Omega))^2$ the orthogonal projections associated to the direct sum (3.10). Applying them to (3.4) and using the identity $P_\nabla \mathbb{M} u = 0$ for all $u \in \mathcal{D}(\mathbb{M})$, the solution to (3.4) thus solves the system:

$$\begin{array}{l} \textit{Find } u = P_0 u + P_\nabla u \in \mathcal{D}(\mathbb{M}) \textit{ such that}: \\ \left\{ \begin{array}{l} P_0 K(p,.)(P_0 u + P_\nabla u) + P_0 \mathbb{M} P_0 u = P_0 f, \\ P_\nabla K(p,.)(P_\nabla u + P_0 u) = P_\nabla f. \end{array} \right. \end{array} \tag{3.11}$$

Second equation of (3.11) is solved with the following lemma:

LEMMA 3.7. *Assume that $(B1)-(B2)-(B3)$ and either $(BT4)$ or $(DT4)$ hold. Then, the operator $P_\nabla K(p,.)P_\nabla \in \mathcal{B}\left(\mathrm{grad}(H_0^1(\Omega))^2\right)$ is invertible with a bounded inverse for all $p \in D_0 \backslash S_0$ where S_0 is a discrete, locally finite and possibly empty set of D_0. Moreover the application $p \in D_0 \backslash S_0 \longmapsto (P_\nabla K(p,.)P_\nabla)^{-1} \in \mathcal{B}\left(\mathrm{grad}(H_0^1(\Omega))^2\right)$ is holomorphic.*

Applying lemma 3.7, the system (3.11) becomes:

$$\begin{array}{l} \textit{Find } u = P_0 u + P_\nabla u \in \mathcal{D}(\mathbb{M}) \textit{ such that}: \\ \left\{ \begin{array}{l} P_\nabla u = (P_\nabla K(p,.)P_\nabla)^{-1} \left[P_\nabla f - P_\nabla K(p,.)P_0 u\right], \\ B(p,.)P_0 u + P_0 \mathbb{M} P_0 u = \widetilde{f}(p), \end{array} \right. \end{array} \tag{3.12}$$

where $\widetilde{f}(p,.) = P_0 f - P_0 K(p,.)P_\nabla \left(P_\nabla K(p,.)P_\nabla\right)^{-1} P_\nabla f$ belongs to $H(\mathrm{div}0,\Omega)^2$, and

$$B(p,.) = P_0 K(p,.)P_0 - P_0 K(p,.)P_\nabla (P_\nabla K(p,.)P_\nabla)^{-1} P_\nabla K(p,.)P_0 \in \mathcal{B}\left(H(\mathrm{div}0,\Omega)^2\right).$$

Let $\widetilde{\mathbb{M}}$ be the restriction of the operator $\mathbb{M}$ to the set $(H(\mathrm{div}0,\Omega))^2$. Then (3.12) is equivalent to the inversion of the holomorphic family of closed operator of $(H(\mathrm{div}0,\Omega))^2$ defined by $\left(B(p,.)+\widetilde{\mathbb{M}}, (H(\mathrm{div}0,\Omega))^2 \cap \mathcal{D}(\mathbb{M})\right)_{p\in D_0\backslash S_0}$. Consider now the following Majda's inequality [24] holding for all $u \in \mathcal{D}(\mathbb{M}) \cap (H(\mathrm{div},\Omega))^2$:

$$\|u\|_{(H^1(\Omega))^6} \le C \left\{ \|u\|_{(L^2(\Omega))^6} + \|\mathbb{M}u\|_{(L^2(\Omega))^6} + \|\mathbb{Q}u\|_{(L^2(\Omega))^6} \right\}, \tag{3.13}$$

where $\mathbb{Q}$ is defined in theorem 3.2. From (3.13), the compactness of the embedding of $H^1(\Omega)$ into $L^2(\Omega)$, and the maximal dissipativity of $(\mathbb{M},\mathcal{D}(\mathbb{M}))$, one obtains that the holomorphic family of closed operators $\left(B(p,.)+\widetilde{\mathbb{M}}, (H(\mathrm{div}0,\Omega))^2 \cap \mathcal{D}(\mathbb{M})\right)_p$ has compact resolvent and non-empty resolvent set for all $p \in D_0\backslash S_0$. From $(BT3)$, there exists p_0 such that $K(p_0,x)$ is coercive for almost all $x \in \Omega$. Then, since the unbounded operator $(\mathbb{M},\mathcal{D}(\mathbb{M}))$ is maximal dissipative, equation (3.4) is well-posed for $p = p_0$. Hence, by the equivalence between (3.4) and (3.12), $B(p_0,.)+\widetilde{\mathbb{M}}$ is invertible on $(H(\mathrm{div}0,\Omega))^2 \cap \mathcal{D}(\mathbb{M})$. Finally, thanks to Fredholm analytic theory $B(p,.)+\widetilde{\mathbb{M}}$ is invertible on $(H(\mathrm{div}0,\Omega))^2 \cap \mathcal{D}(\mathbb{M})$ for all $p \in D_0\backslash S$, where $S = S_0 \cup S_1$ for a discrete, locally finite and possibly empty set of $D_0\backslash S_0$ noted S_1.

Remark that the inverse operator $\left(B(p,.)+\widetilde{\mathbb{M}}\right)^{-1}$ belongs to $\mathcal{B}(H(\mathrm{div}0,\Omega)^2)$ and that the application $p \in D_0\backslash S \longmapsto \left(B(p,.)+\widetilde{\mathbb{M}}\right)^{-1}$ is holomorphic. Thus $P_0 u(p,.)$ is holomorphic too. Then, from Lemma 3.7 we derive that $P_\nabla u(p,.)$ is holomorphic on $D_0\backslash S$ and so is the application $p \in D_0\backslash S \longmapsto u(p,.) = P_\nabla u(p,.) + P_0 u(p,.) \in L^2(\Omega)^6$. Finally we have the bound:

$$\|u(p,.)\|_{L^2(\Omega)} \le \|P_\nabla u(p,.)\|_{L^2(\Omega)} + \|P_0 u(p,.)\|_{L^2(\Omega)} \le C(p)\, \|f\|_{L^2(\Omega)},$$

where $C(p)$ is a positive constant depending only on Ω and p. □

The rest of this section is dedicated to prove the lemmas.

Proof. [**Proof of lemma 3.7**] We begin by proving the lemma when $(BT1)-(BT2)-(BT3)-(BT4)$ hold. Inverting $P_\nabla K(p,.)P_\nabla$ on $\mathrm{grad}(H_0^1(\Omega))^2$ summarizes as solving the problem:

$$\begin{array}{l} Find\ \varphi \in H_0^1(\Omega)^2\ such\ that,\ for\ all\ v \in H_0^1(\Omega)^2 : \\ \int_\Omega \langle K(p,x)\mathbb{G}\varphi, \overline{\mathbb{G}v}\rangle\, dx = < h, v >_{H^{-1}(\Omega)^2\times H_0^1(\Omega)^2}, \end{array} \tag{3.14}$$

where h belongs to $H^{-1}(\Omega)^2$ and $\mathbb{G}\varphi = (-\nabla\varphi_1, -\nabla\varphi_2)$ for $\varphi = (\varphi_1,\varphi_2) \in H_0^1(\Omega)^2$. System (3.14) is equivalent to the following second order partial differential equation:

$$\begin{array}{c} Find\ \varphi \in H_0^1(\Omega)^2\ such\ that : \\ -\mathbb{Q}\left(K(p,.)\mathbb{G}\varphi(p,x)\right) = h. \end{array} \tag{3.15}$$

Unfortunately, the principal part of this second order partial differential operator fails to be coercive for some $p \in D_0$. However, introducing the following change of unknown:

$$\varphi(p,.) = a(p,.)\psi(p,.), \tag{3.16}$$

where a is given by $(BT4)$. Reporting (3.16) into (3.15) leads to an equivalent in ψ:

$$\begin{array}{l} Find\ \psi \in H_0^1(\Omega)^2\ such\ that : \\ \mathbb{S}(p)\psi := -\mathbb{Q}K(p,x)\left(a(p,x)\mathbb{G}\psi(p,x) + \begin{pmatrix} \psi_1(p,x)\nabla a(p,x) \\ \psi_2(p,x)\nabla a(p,x) \end{pmatrix}\right) = h. \end{array} \tag{3.17}$$

From $(BT4)$ the multiplicative operator $a(p,.)K(p,.)$ is coercive for all $p \in D_0$ implying that the principal part of (3.17) is coercive. Moreover, $a(p,.)$ is Lipschitz continuous for all $p \in D_0$ so Rademacher's theorem shows that $\nabla a(p,.)$ belongs to $L^\infty(\Omega)$. Thus, $\mathbb{S}(p) : H_0^1(\Omega)^2 \to H^{-1}(\Omega)^2$ is a holomorphic family of closed operator.

Let assume now that there exists λ in the resolvent set of $\left(\mathbb{S}(p), H_0^1(\Omega)^2\right)$. Since the embedding of $H_0^1(\Omega)$ into $H^{-1}(\Omega)$ is compact, the resolvent operator $(\mathbb{S}(p)-\lambda)^{-1}$ defines a compact operator of $H^{-1}(\Omega)^2$. Consequently, solving equation (3.17) is the same as inverting a holomorphic family of closed operators with compact resolvent. This is achieved with help of the Fredholm analytic theory since we show that the resolvent set of $\left(\mathbb{S}(p), H_0^1(\Omega)^2\right)$ is non-empty for all $p \in D_0$. Thus, let λ be an arbitrary complex number and $A_p(\psi, v)$ be given, for any $\psi, v \in H_0^1(\Omega)^2$, by

$$A_p(\psi,v) = \int_\Omega \langle K(p,x)a(p,x)\mathbb{G}\psi, \overline{\mathbb{G}v}\rangle + \left\langle K(p,x)\begin{pmatrix} \psi_1(p,x)\nabla a(p,x) \\ \psi_2(p,x)\nabla a(p,x) \end{pmatrix}, \overline{\mathbb{G}v}\right\rangle dx.$$

Hypothesis $(BT4)$ together with Cauchy-Schwartz inequality imply then the bound:

$$\begin{aligned} \mathcal{R}e(A_p(\psi,\psi)) + \mathcal{R}e(\lambda)\int_\Omega |\psi|^2 dx \ \geq\ & \alpha \parallel \mathbb{G}\psi \parallel^2_{L^2(\Omega)} + \mathcal{R}e(\lambda)\,\|\psi\|^2_{L^2(\Omega)} \\ & - 2\|K(p,.)\mathbb{G}a(p,.)\|_{L^\infty(\Omega)}\,\|\psi\|_{L^2(\Omega)^2}\,\|\mathbb{G}\psi\|_{L^2(\Omega)^2}\,. \end{aligned}$$

Using Young inequality $ab \leq \xi a^2/2 + b^2/(2\xi)$ onto the term $\|\psi\|_{L^2(\Omega)^2} \|\mathbb{G}\psi\|_{L^2(\Omega)^2}$, with $\xi = \alpha/\left(2\,\|K(p,.)\mathbb{G}a(p,.)\|_{L^\infty(\Omega)}\right)$ and, gathering the previous calculations yields:

$$\mathcal{R}e(A_p(\psi,\psi)) + \mathcal{R}e(\lambda)\int_\Omega |\psi|^2 dx \geq \left(\mathcal{R}e(\lambda) - \frac{\|K(p,.)\mathbb{G}a(p,.)\|^2_{L^\infty(\Omega)}}{\alpha}\right)\|\psi\|^2_{L^2(\Omega)^2} + \frac{\alpha}{2}\|\mathbb{G}\psi\|^2_{L^2(\Omega)}.$$

Thus, taking $\lambda \in \mathbb{C}$ with a big enough real part and using Lax-Milgram theorem, we obtain that the resolvent set of $\left(\mathbb{S}(p), H_0^1(\Omega)^2\right)$ is non-empty for all $p \in D_0$. It remains to find p such that equation (3.17) is well-posed. Taking $p = p_0$ defined through assumption $(BT3)$ we get that (3.14) is well-posed and the reverse change of unknown (3.16) implies that (3.17) is well-posed too for $p = p_0$. Hence the holomorphic family of closed operator $\left(\mathbb{S}(p), H_0^1(\Omega)^2\right)$ is invertible for all $p \in D_0\backslash S_0$ where S_0 is a discrete, locally finite and possibly empty set of D_0. Moreover, the inverse operator $\mathbb{S}(p)^{-1}$ which belong to $\mathcal{B}(H^{-1}(\Omega)^2, H_0^1(\Omega)^2)$ is holomorphic for all $p \in D_0\backslash S_0$ because the application $p \in D_0 \longmapsto \mathbb{S}(p) \in \mathcal{B}(H_0^1(\Omega)^2, H^{-1}(\Omega)^2)$ is holomorphic ([19] p.365).

Since equations (3.17) and (3.14) are equivalent, we derive, from the well-posedness of (3.17), the invertibility of the operator $P_\nabla K(p,.)P_\nabla$ on $(\mathrm{grad}(H_0^1(\Omega)))^2$ for all $p \in D_0\backslash S_0$. Moreover, the closed graph theorem shows that $(P_\nabla K(p,.)P_\nabla)^{-1}$ actually belongs to $\mathcal{B}\left(\mathrm{grad}(H_0^1(\Omega))^2\right)$. Finally, remark that the operator $(P_\nabla K(p,.)P_\nabla)^{-1} \in \mathcal{B}\left(\mathrm{grad}(H_0^1(\Omega))^2\right)$ is holomorphic for all $p \in D_0\backslash S_0$ since the application $p \in D_0 \longmapsto P_\nabla K(p,.)P_\nabla \in \mathcal{B}\left(\mathrm{grad}(H_0^1(\Omega))^2\right)$ is holomorphic ([19] p.361).

When $(DT4)$ holds instead of $(BT4)$, the tensor $K(p,x)$ is the block-diagonal matrix $K(p,x) = p\,\mathrm{diag}\,(\varepsilon(p,x), \mu(p,x))$. Hence, equation (3.14) reduces to

$$\begin{array}{l} Find\ \varphi \in H_0^1(\Omega)\ such\ that: \\ -\mathrm{div}\,(p\vartheta(p,x)\nabla\varphi_\vartheta) = h_\vartheta, \end{array}$$

for $h_\vartheta \in H^{-1}(\Omega)$ and $\vartheta \in \{\varepsilon, \mu\}$. Introducing the changes of unknown $\varphi_\vartheta(p,.) = a_\vartheta(p,.)\psi_\vartheta(p,.)$, where a_ϑ is given in $(DT4)$, yields to

$$\begin{array}{l} Find\ \psi \in H_0^1(\Omega)\ such\ that: \\ -\mathrm{div}\,(\vartheta(p,x)a_\vartheta(p,x)\nabla\psi_\vartheta(p,x)) - \mathrm{div}\,(\vartheta(p,x)\psi_\vartheta(p,x)\nabla a_\vartheta(p,x)) = h_\vartheta,\ \text{on } \Omega. \end{array}$$

From now, the proof of the existence and uniqueness of ψ_ϑ can be achieved, in the same way as before, using the Fredholm analytic theory once for each $\vartheta \in \{\varepsilon, \mu\}$. □

4. Acoustic wave system. In this section, we are investigating the first order wave equation:

$$\begin{array}{l} Find\ (u,\rho) \in\ H(\mathrm{div},\Omega) \times H^1(\Omega)\ such\ that: \\ \left\{ \begin{array}{l} \Gamma(p,x)u(p,x) - \nabla\rho(p,x) = f_1(x),\ \text{on } \Omega, \\ n(p,x)\rho(p,x) - \mathrm{div}(u(p,x)) = f_2(x),\ \text{on } \Omega, \\ \rho(p,x) + \lambda(x)\langle u(p,x), \nu\rangle = 0,\ \text{on } \partial\Omega, \end{array} \right. \end{array} \tag{4.1}$$

where ρ denotes the acoustic velocity, u is the acoustic pressure, $f = (f_1, f_2) \in L^2(\Omega)^3 \times L^2(\Omega)$ is a source term, λ is the acoustic impedance, Γ is the bulk modulus and n is the refractive index. The normal trace of $u \in H(\mathrm{div},\Omega)$ appearing in the boundary conditions has to be understood in the sense that $\langle u,\nu\rangle_{|\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega)$. Finally, assume that the function $\lambda : \partial\Omega \mapsto \mathbb{C}$ is Lipschitz continuous with $\mathcal{R}e(\lambda) \geq 0$.

We introduce the following conditions to be satisfied by the material:

Assumptions 8 (For acoustic materials).

(A1) *The applications $\Gamma(p,x)$ and $n(p,x)$ are holomorphic in p on D_0 for almost all $x \in \Omega$, where D_0 is a connected open set of $\mathbb{C}$.*

(A2) *The applications $\Gamma(p,x)$ and $n(p,x)$ both belong to $L^\infty(\overline{\Omega})$ and are invertible for all $p \in D_0$.*

(A3) *There exists p_0 in D_0 and $\alpha > 0$ such that the following inequality holds:*

$$\mathcal{R}e\left\{< \Gamma(p_0,x)X, \overline{X} >\right\} + |z|^2 \mathcal{R}e(n(p_0,x)) \geq \alpha(|X|^2 + |z|^2),$$

for all $(X,z) \in \mathbb{C}^3 \times \mathbb{C}$ and for almost all $x \in \Omega$.

Our first generic well-posedness result for (4.2) is given below:

Theorem 4.1. *Suppose that $\Gamma(p) \in \mathbb{C}\backslash\{0\}$ is scalar valued and does not depend on x, and assume that assumption 8 is fulfilled. Then equation (4.2) is well-posed for all $p \in D_0\backslash S$ where $S \subset D_0$ is a discrete, locally finite and possibly empty set of D_0. Moreover, the solution is holomorphic from $D_0\backslash S$ to $L^2(\Omega)^4$ and continuous with respect to the data.*

Proof. Firstly, rewrite (4.1) as the following general first order wave equation:

$$\begin{aligned}&\textit{Find } (u,\rho) \in \mathcal{H}_{\mathbb{W}} \textit{ such that}:\\ &\left\{\begin{array}{l} (K(p,x) - \mathbb{W})\begin{pmatrix} u \\ \rho \end{pmatrix} = f(x),\ x \in \Omega, \\ \rho(p,x) + \lambda(x)\langle u(p,x),\nu\rangle = 0,\ x \in \partial\Omega, \end{array}\right.\end{aligned} \tag{4.2}$$

where $f \in L^2(\Omega)^4$ and $K(p,x) = \begin{pmatrix} \Gamma(p) & 0 \\ 0\mathbb{I}_3 & n(p,x) \end{pmatrix}$. The operator $\mathbb{W}$ is defined by:

$$\mathbb{W} = \begin{pmatrix} 0_{3\times3} & \nabla \\ \mathrm{div} & 0 \end{pmatrix},$$

whose domain is given below:

$$\mathcal{D}(\mathbb{W}) = \left\{(u,\rho) \in H(\mathrm{div},\Omega) \times H^1(\Omega)\ |\rho(x) + \lambda(x)\langle u(x),\nu\rangle = 0,\ x \in \partial\Omega\right\}.$$

The operator $(-\mathbb{W},\mathcal{D}(\mathbb{W}))$ is maximal dissipative [31]. Now consider the following Hodge decomposition ([4] p. 54, theorem 10):

$$L^2(\Omega)^3 = \mathrm{grad}H^1(\Omega) \oplus \left(\nabla \times \widetilde{V}\right), \tag{4.3}$$

where $\widetilde{V} = \left\{\Psi \in H(\mathrm{curl},\Omega)\mid \mathrm{div}\Psi = 0,\ \nu \times \Psi_{|\partial\Omega} = 0\right\}$, and introduce the orthogonal projections associated to the direct sum (4.3), $P_0 : L^2(\Omega)^6 \longrightarrow \nabla \times \widetilde{V}$ and $P_\nabla : L^2(\Omega)^6 \longrightarrow \mathrm{grad}H^1(\Omega)$. Noting that $P_0\nabla\phi = 0$ for all $\phi \in H^1(\Omega)$, (4.1) becomes:

$$\begin{aligned}&\textit{Find } (u,\rho) = (P_0 v + P_\nabla u,\rho) \in \mathcal{D}(\mathbb{W}) \textit{ such that}:\\ &\left\{\begin{array}{l} P_0\Gamma(p)P_0 u + P_0\Gamma(p)P_\nabla u = P_0 f_1,\ x \in \Omega, \\ P_\nabla\Gamma(p)P_\nabla u + P_\nabla\Gamma(p)P_0 u - \nabla\rho = P_\nabla f_1,\ x \in \Omega, \\ n(p,x)\rho - \mathrm{div}(P_\nabla u) = f_2\ x \in \Omega. \end{array}\right.\end{aligned} \tag{4.4}$$

Lemma 4.2. *The operator $P_0\Gamma(p)P_0 \in \mathcal{B}\left(\nabla\times\widetilde{V}\right)$ is invertible with a bounded inverse for all $p \in D_0$. Moreover the application $p \in D_0 \longmapsto (P_0\Gamma(p)P_0)^{-1} \in \mathcal{B}\left(\nabla\times\widetilde{V}\right)$ is holomorphic.*

Lemma 4.2 shows that $P_0u = (P_0\Gamma(p,.)P_0)^{-1}\{-P_0\Gamma(p,.)P_\triangledown u + P_0 f_1\}$ implying that one can rewrite system (4.4) into the form:

$$\begin{array}{c} Find\ (w,\rho) \in \mathcal{D}(\widetilde{\mathbb{W}})\ such\ that: \\ \left(B(p,.) - \widetilde{\mathbb{W}}\right)\begin{pmatrix} w \\ \rho \end{pmatrix} = g, \end{array} \tag{4.5}$$

where $\widetilde{\mathbb{W}} = \begin{pmatrix} 0_{3\times 3} & \nabla \\ \mathrm{div}P_\triangledown & 0 \end{pmatrix}$ with domain $\mathcal{D}(\widetilde{\mathbb{W}}) = \left\{(u,\rho) \in \mathcal{D}(\mathbb{W})\ |u \in \mathrm{grad}H^1(\Omega)^3\right\}$, and $w = P_\triangledown u$, $g \in \mathrm{grad}H^1(\Omega)\times L^2(\Omega)$ and $B(p,.) \in \mathcal{B}\left(L^2(\Omega)^4\right)$ is defined by:

$$B(p,.) = \begin{pmatrix} P_\triangledown\Gamma(p)P_\triangledown - P_\triangledown\Gamma(p)P_0\left(P_0\Gamma(p)P_0\right)^{-1}P_0\Gamma(p)P_\triangledown & 0 \\ 0 & n(p,.) \end{pmatrix}.$$

From $(A1)-(A2)$ and lemma 4.2, the function $p \mapsto B(p)$ is holomorphic on D_0.

From [24], we have for all $(u,\rho) \in \mathcal{D}(\mathbb{W}) \cap (H(\mathrm{curl},\Omega)\times L^2(\Omega))$:

$$\|(u,\rho)\|_{H^1(\Omega)^4} \le C\left\{\|(u,\rho)\|_{L^2(\Omega)^4} + \|\mathbb{W}(u,\rho)\|_{L^2(\Omega)^4} + \|\nabla\times u\|_{L^2(\Omega)^3}\right\}. \tag{4.6}$$

Then the identity $\nabla\times\nabla\phi = 0$ and (4.6) show that the operator $\left(\widetilde{\mathbb{W}}, \mathcal{D}(\widetilde{\mathbb{W}})\right)$ has compact resolvent. Consequently, solving (4.5) is the same as inverting the holomorphic family of closed operators with compact resolvent, having non-empty resolvent set, given by $\left(B(p,.) - \widetilde{\mathbb{W}}, \mathcal{D}(\widetilde{\mathbb{W}})\right)_{p\in D_0}$. The later is done with Fredholm analytic theory. Using $(A3)$, there exists p_0 such that $K(p_0,.)$ is coercive. The unbounded operator $(-\mathbb{W}, \mathcal{D}(\mathbb{W}))$ is maximal dissipative so (4.2) is well-posed for $p = p_0$ and the equivalence between problems (4.2) and (4.5) shows that $B(p_0,.) - \widetilde{\mathbb{W}}$ is invertible on $\mathcal{D}(\widetilde{\mathbb{W}})$. Hence we obtain that the operator $B(p,.) - \widetilde{\mathbb{W}}$ is invertible on $\mathcal{D}(\widetilde{\mathbb{W}})$ for all $p \in D_0\backslash S$, where S is a discrete, locally finite and possibly empty set of D_0.

Finally, remark that the inverse operator $\left(B(p,.) - \widetilde{\mathbb{W}}\right)^{-1} \in \mathcal{B}(\mathrm{grad}H^1(\Omega)\times L^2(\Omega))$ and that the application

$$p \in D_0\backslash S \longmapsto \left(B(p,.) - \widetilde{\mathbb{W}}\right)^{-1} \in \mathcal{B}\left(\mathrm{grad}H^1(\Omega)\times L^2(\Omega)\right),$$

is holomorphic. From Lemma 4.2 we then derive that $(P_0v(p,.),\rho)$ is holomorphic on $D_0\backslash S$ and hence the application $p \in D_0\backslash S \longmapsto (P_\triangledown v(p,.) + P_0v(p,.), \rho(p,.)) \in L^2(\Omega)^4$ is holomorphic too. Moreover, using the fact that $u(p,.) = a(p,.)v(p,.)$, it comes the holomorphy of $p \in D_0\backslash S \longmapsto (u(p,.),\rho(p,.)) \in L^2(\Omega)^4$ and the bound:

$$\forall p \in D_0\backslash S, \|(u(p,.),\rho(p,.))\|_{L^2(\Omega)^4} \le C(p)\|f\|_{L^2(\Omega)^4}.$$

This concludes the proof. □

Before proving lemma 4.2, we need the following technical result:

Lemma 4.3. *There exists a constant $C > 0$ depending only on Ω such that:*

$$\|\Phi\|_{L^2(\Omega)^3} \leq C \, \|\nabla \times \Phi\|_{L^2(\Omega)^3} \,, \text{ for all } \Phi \in \widetilde{V}. \tag{4.7}$$

Proof. [**Proof of lemma 4.3**] First, one has for all $\Phi \in \widetilde{V}$ [9, 24]

$$\|\Phi\|_{(H^1(\Omega))^3} \leq C \left\{ \|\Phi\|_{(L^2(\Omega))^3} + \|\nabla \times \Phi\|_{(L^2(\Omega))^3} + \|\mathrm{div}(\Phi)\|_{L^2(\Omega)} \right\}. \tag{4.8}$$

Inequality (4.7) is then established by contradiction. Let $(\Phi_n)_n \subset \widetilde{V}$ be a sequence such that $\|\Phi_n\|_{L^2(\Omega)^3} = 1$ and $\|\nabla \times \Phi_n\|_{L^2(\Omega)^3} \leq \frac{1}{n}$. From (4.8) $(\Phi_n)_n$ is bounded in H^1 norm, hence it has a strongly convergent subsequence in the $L^2(\Omega)^3$ norm toward some $\Phi_0 \in H^1(\Omega)^3$. Since $\Phi_n \in \widetilde{V}$, for all n, one obtains $\mathrm{div}\Phi_0 = 0$ in the sense of distributions and $\nu \times \Phi_{0|\partial\Omega} = 0$ in $H^{-\frac{1}{2}}(\partial\Omega)^3$. Moreover, as $\|\nabla \times \Phi_n\|_{L^2(\Omega)^3} \leq \frac{1}{n}$ it comes $\nabla \times \Phi_0 = 0$. Consider now the following Hodge decomposition ([9] p.353):

$$L^2(\Omega)^3 = H_0(\mathrm{curl}0, \Omega) \oplus H(\mathrm{div}0, \Omega),$$

where $H_0(\mathrm{curl}0, \Omega) = \left\{ v \in L^2(\Omega)^3 \mid \nabla \times v = 0, \text{ on } \Omega, \ \nu \times v = 0, \text{ on } \partial\Omega \right\}$. We thus deduce that Φ_0 both belongs to $H_0(\mathrm{curl}0, \Omega)$ and $H(\mathrm{div}0, \Omega)$ so $\Phi_0 = 0$ which contradicts the fact that $\|\Phi_0\|_{L^2(\Omega)^3} = 1$. □

Remark 4.4. *Lemma 4.3 is recovered from inequality (4.8) by compactness but holds without (4.8) (see p.553 in [13]).*

Proof. [**Proof of lemma 4.2**] First note that inverting $P_0\Gamma(p)P_0$ on $\nabla \times \widetilde{V}$ is the same as solving the following boundary value problem:

$$\begin{aligned} &Find \ \Phi \in \widetilde{V} \ such \ that, \ for \ all \ \Psi \in \widetilde{V}: \\ &\int_\Omega \left\langle \Gamma(p) \nabla \times \Phi, \overline{\nabla \times \Psi} \right\rangle dx = < h, \Psi >_{\widetilde{V}' \times \widetilde{V}}, \end{aligned} \tag{4.9}$$

where h belongs to $\widetilde{V}'$, which is the set of continuous linear forms on $\widetilde{V}$. Integrating by parts and using the boundary condition, Φ is thus solution to the equation:

$$\begin{aligned} &Find \ \Phi \in \widetilde{V} \ such \ that: \\ &\left\{ \begin{array}{l} \nabla \times \nabla \times \Phi = h(p, .), \text{ in } \Omega \\ \nu \times \Phi_{|\partial\Omega} = 0, \text{ on } \partial\Omega, \end{array} \right. \end{aligned}$$

where $h(p) = h/\Gamma(p) \in \widetilde{V}'$. Note, from inequality (4.8), that the set $\widetilde{V}$ is a Hilbert space when equipped with the usual $H^1(\Omega)^3$ norm. The coercivity of the bilinear form involved in (4.9) thus follows from lemma 4.3 and the proof of the first part of the lemma 4.2 is then done by using Lax-Milgram lemma. The holomorphy of $p \in D_0 \longmapsto (P_0\Gamma(p)P_0)^{-1} \in \mathcal{B}\left(\nabla \times \widetilde{V}\right)$ then comes from [19] and the holomorphy of $\Gamma(p)$. □

The range of application of the result presented into theorem 4.1 seems to be quite restricted since it requires the physical parameters to not depend on x. However it can be extended with following assumption:

Assumptions 9 (For non-constant and tensorial acoustic materials). *Suppose that $(A1) - (A2) - (A3)$ and $(A4)$ hold, with the additional condition*

(A4) There exists $a(p,x) \in \mathbb{C}$, Lipschitz continuous for $x \in \overline{\Omega}$ and holomorphic for $p \in D_0$, satisfying $a(p,.)_{|\partial\Omega} = 1$ and such that the following inequality holds:

$$\mathcal{R}e \left\langle \Gamma(p,x) a(p,x) X, \overline{X} \right\rangle \geq \alpha |X|^2,$$

for all $p \in D_0$, for almost all $x \in \Omega$ and for all $X \in \mathbb{C}^3$.

Corollary 4.5. *If assumption 9 is satisfied, then equation (4.1) is well-posed for all $p \in D_0 \backslash S$ where $S \subset D_0$ is a discrete, locally finite and possibly empty set of D_0. Moreover, the solution satisfies the bound:*

$$\forall p \in D_0 \backslash S, \|(u(p,.), \rho(p,.))\|_{L^2(\Omega)^4} \leq C(p) \|f\|_{L^2(\Omega)^4},$$

and the application $p \in D_0 \backslash S \longmapsto (u(p,.), \rho(p,.)) \in L^2(\Omega)^4$ is holomorphic.

Proof. We follow the proof of theorem 4.1. However, before projecting equation (4.2) according to Hodge decomposition (4.3), we perform the change of unknown $u = a(p,.)v$. Then one derives from $(A4)$ that (v, ρ) belongs to $\mathcal{D}(\mathbb{W})$ and is solution to:

$$\begin{aligned} &Find\ (v,\rho) \in \mathcal{D}(\mathbb{W})\ such\ that:\\ &\begin{cases} \Gamma(p,.)a(p,x)v(p,x) - \nabla\rho(p,x) = f_1(x),\ x \in \Omega, \\ a^{-1}(p,x)n(p,x)\rho(p,x) + a(p,x)^{-1} < \nabla a(p,x), v(p,x) > \\ \qquad\qquad -\mathrm{div}(v(p,x)) = a^{-1}(p,x) f_2(x),\ \text{for } x \in \Omega. \end{cases} \end{aligned} \tag{4.10}$$

Then projecting (4.10), it comes:

$$\begin{aligned} &Find\ (v,\rho) = (P_0 v + P_\nabla v, \rho) \in \mathcal{D}(\mathbb{W})\ such\ that:\\ &\begin{cases} P_0\Gamma(p,x)a(p,x)P_0 v + P_0\Gamma(p,x)a(p,x)P_\nabla v = P_0 f_1,\ x \in \Omega, \\ P_\nabla\Gamma(p,x)a(p,x)P_\nabla v + P_\nabla\Gamma(p,x)a(p,x)P_0 v - \nabla\rho = P_\nabla f_1,\ x \in \Omega, \\ a^{-1}(p,x)n(p,x)\rho(p,x) + a(p,x)^{-1} < \nabla a(p,x), v(p,x) > \\ \qquad\qquad -\mathrm{div}(P_\nabla v) = a^{-1}(p,x) f_2(x),\ x \in \Omega. \end{cases} \end{aligned} \tag{4.11}$$

Lemma 4.6. *The operator $P_0\Gamma(p,.)a(p,.)P_0 \in \mathcal{B}\left(\nabla \times \widetilde{V}\right)$ is invertible with a bounded inverse for all $p \in D_0$ and the application $p \in D_0 \longmapsto (P_0\Gamma(p,.)a(p,.)P_0)^{-1} \in \mathcal{B}\left(\nabla \times \widetilde{V}\right)$ is holomorphic.*

Lemma 4.6 shows that $P_0 v = (P_0\Gamma(p,.)a(p,.)P_0)^{-1} \{-P_0\Gamma(p,.)a(p,.)P_\nabla v + P_0 f_1\}$ implying system (4.11) to be rewritten into the form:

$$\begin{aligned} &Find\ (P_\nabla v, \rho) \in \mathcal{D}(\widetilde{\mathbb{W}})\ such\ that:\\ &\left(B(p,.) - \widetilde{\mathbb{W}}\right) \begin{pmatrix} P_\nabla v \\ \rho \end{pmatrix} = g,\ \text{on } \Omega, \end{aligned}$$

where $g \in \mathrm{grad} H^1(\Omega) \times L^2(\Omega)$, $B(p,.) \in \mathcal{B}\left(L^2(\Omega)^4\right)$ is a bounded operator which is holomorphic on D_0 (thanks to $(A1) - (A2) - (A4)$ and lemma 4.6). The end of the proof is now the same as for theorem 4.1. □

Proof. [**Proof of lemma 4.6**] Following the proof of lemma 4.2, it comes that inverting $P_0\Gamma(p,.)a(p,.)P_0$ on $\nabla \times \widetilde{V}$ remains to find $\Phi \in \widetilde{V}$ solution to:

$$Find\ \Phi \in \widetilde{V}\ such\ that: \\ \begin{cases} \nabla \times \Gamma(p,x)a(p,x)\nabla \times \Phi(p,x) = h,\ x \in \Omega, \\ \nu \times \Phi(p,x) = 0,\ \text{on } \partial\Omega. \end{cases}$$

From assumption $(A4)$, it follows for all $p \in D_0$ that the multiplicative operator $a(p,.)\Gamma(p,.)$ is coercive. The bilinear form $(\Phi,\Psi) \longmapsto \int_\Omega \left\langle \Gamma(p,.)a(p,.)\nabla \times \Phi, \overline{\nabla \times \Psi} \right\rangle dx$ is then coercive on $\widetilde{V}$ equipped with the usual $H^1(\Omega)^3$ norm (see lemma 4.3). Consequently Lax-Milgram theorem shows the first part of the lemma. The holomorphy of $p \in D_0 \longmapsto (P_0\Gamma(p,.)a(p,.)P_0)^{-1} \in \mathcal{B}\left(\nabla \times \widetilde{V}\right)$ comes from the holomorphy of $\Gamma(p,.)a(p,.)$ (thanks to $(A1)-(A2)-(A4)$) [19]. □

5. Generic well-posedness for linear elasticity. The physical properties of the elastic metamaterial manifest with for instance negative mass density or bulk modulus [10, 36]. Hence, we consider the equations of elastodynamics [6]:

$$Find\ u = (u_1,u_2,u_3) \in H^1(\Omega)^3\ such\ that\ for\ j=1,2,3: \\ \begin{cases} \operatorname{div}(\mu(p,x)\nabla u_j(p,x)) + \partial_j(\lambda(p,x)+\mu(p,x))\operatorname{div}(u(p,x)) \\ \qquad\qquad -p^2 u_j(p,x) = f(x),\ \text{on } \Omega, \\ \left\langle (\mu(p,x)\nabla u_j)_{|\partial\Omega}, \nu \right\rangle + \nu_j(\lambda(p,x)+\mu(p,x))_{|\partial\Omega}\operatorname{div}(u)_{|\partial\Omega} \\ \qquad\qquad -(\Lambda(x)u_j)_{|\partial\Omega} = 0,\ \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where λ, μ are the Lamé coefficients, f is the body force per unit volume, $\Lambda : \partial\Omega \longrightarrow \mathbb{C}$ is an impedance assumed to be coercive and u is the displacement. To formulate (5.1) as a first order system of partial differential equation, we introduce new unknowns:

$$\begin{cases} v_j = \mu(p,x)\nabla u_j,\ j=1,2,3, \\ \gamma = (\lambda(p,x)+\mu(p,x))\operatorname{div} u. \end{cases}$$

Thus for $j = 1,2,3$, (5.1) reduces to the following first order system:

$$Find\ (v_j,u_j,\gamma) \in H(\operatorname{div},\Omega) \times H^1(\Omega) \times H^1(\Omega)\ such\ that: \\ \begin{cases} p^2 u_j - \operatorname{div}(v_j) - \partial_j\gamma = -f,\ \text{on } \Omega, \\ \mu^{-1}(p,x)v_j - \nabla u_j = 0,\ \text{on } \Omega, \\ (\lambda(p,x)+\mu(p,x))^{-1}\gamma - \operatorname{div}(u) = 0,\ \text{on } \Omega, \\ < v_j(x), \nu > + \nu_j\gamma(x) - \Lambda(x)u_j(x) = 0,\ \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

The conditions to be verified by the metamaterial are thus:

Assumptions 10 (For elastic materials).

(E1) The applications $\lambda(p,x)^{-1}$ and $(\lambda(p,x)+\mu(p,x))^{-1}$ are holomorphic on D_0 for almost all $x \in \Omega$ where D_0 is a connected open set of $\mathbb{C}$.

(E2) The applications $\lambda(p,x)^{-1}$ and $(\lambda(p,x)+\mu(p,x))^{-1}$ belong to $L^\infty(\overline{\Omega})$ for all $p \in D_0$.

(E3) There exists p_0 in D_0 and $\alpha > 0$ such that the following inequality holds:

$$\mathcal{R}e\left\{< p_0^2 X, \overline{X} > + \mathcal{R}e < \mu^{-1}(p_0,x)Y, \overline{Y} >\right\} \\ + \mathcal{R}e(\lambda(p_0,x)+\mu(p_0,x))^{-1}|z|^2 \geq \alpha(|X|^2+|Y|^2+|z|^2),$$

for all $(X,Y,z) \in \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}$ and for almost all $x \in \Omega$.

We then have the result:

THEOREM 5.1. *Suppose that $\mu(p,.) \neq 0$ (see (5.2)) is scalar valued and does not depend on x. If assumption 10 is fulfilled, then system (5.2) is well-posed for all $p \in D_0 \backslash S$ where $S \subset D_0$ is a discrete, locally finite and possibly empty set of D_0. Moreover, the solution is continuous with respect to the data and the application $p \in D_0 \backslash S \longmapsto (v(p,.), u(p,.), \gamma(p,.)) \in L^2(\Omega)^{13}$ is holomorphic.*

Proof. First of all, let us introduce the following unbounded operator:

$$\mathbb{E} = \begin{pmatrix} 0 & 0 & 0 & \nabla & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \nabla & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \nabla & 0 \\ \mathrm{div} & 0 & 0 & 0 & 0 & 0 & \partial_1 \\ 0 & \mathrm{div} & 0 & 0 & 0 & 0 & \partial_2 \\ 0 & 0 & \mathrm{div} & 0 & 0 & 0 & \partial_3 \\ 0 & 0 & 0 & \partial_1 & \partial_2 & \partial_3 & 0 \end{pmatrix},$$

acting on vector fields of the form $(v_1, v_2, v_3, u_1, u_2, u_3, \gamma) \in \mathbb{C}^{13}$ on the domain $\mathcal{D}(\mathbb{E})$

$$\mathcal{D}(\mathbb{E}) = \left\{ \forall j,\, (v_j, u_j, \gamma) \in H(\mathrm{div}, \Omega) \times H^1(\Omega) \times H^1(\Omega) \middle|\, (v_j, u_j, \gamma)|_{\partial\Omega} \in \ker \widetilde{N}_j(x) \right\},$$

where $\widetilde{N}_j(x)(v_j, u_j, \gamma) = < (v_j(x), \nu > + \nu_j \gamma(x) - \Lambda u_j(x)$. Thus (5.2) collapses as

$$\begin{array}{l} \textit{Find } \Pi = (v_1, v_2, v_3, u_1, u_2, u_3, \gamma) \in \mathcal{H}_{\mathbb{E}}, \textit{ such that}: \\ \quad \left\{ \begin{array}{l} (K(p,x) - \mathbb{E})\, \Pi = F, \; x \in \Omega, \\ < (v_j)_{|\partial\Omega}, \nu > + \nu_j \gamma_{|\partial\Omega} - (\Lambda(x) u_j)_{|\partial\Omega} = 0, \text{ on } \partial\Omega, \end{array} \right. \end{array} \tag{5.3}$$

where $F \in L^2(\Omega)^{13}$ and the multiplicative operator is given by

$$K(p,x) = \begin{pmatrix} \mu^{-1}(p)\mathbb{I}_9 & 0\mathbb{I}_3 & 0 \\ 0\mathbb{I}_9 & p^2\mathbb{I}_3 & 0 \\ 0\mathbb{I}_9 & 0\mathbb{I}_3 & (\lambda(p,x) + \mu(p,.))^{-1} \end{pmatrix}. \tag{5.4}$$

The operator $(-\mathbb{E}, \mathcal{D}(\mathbb{E}))$ being maximal dissipative, (5.3) is well-posed for $p = p_0$ [31].

At last, (5.2) is similar to the first order wave equation (4.1). Hence, one simply needs to control the curl of vector fields $v_j \in H(\mathrm{div}, \Omega)$ to recover some compactness for the resolvent of $(\mathbb{E}, \mathcal{D}(\mathbb{E}))$. Thus, using the results of [24], the following inequality holds for all $\Pi = (v_1, v_2, v_3, u_1, u_2, u_3, \gamma) \in \mathcal{D}(\mathbb{E}) \cap (H(\mathrm{curl}, \Omega))^3 \times L^2(\Omega)^4$:

$$\|\Pi\|_{H^1(\Omega)^{13}} \leq C \left(\|\Pi\|_{L^2(\Omega)^{13}} + \|\mathbb{E}\Pi\|_{L^2(\Omega)^{13}} + \sum_{j=1}^{3} \|\nabla \times v_j\|_{L^2(\Omega)^3} \right).$$

From now, we mimic the proof of theorem 4.1. First, projecting equation (5.3) with help of Hodge decomposition (4.3), and using the same notations as for the acoustics, we infer that:

$$P_0 v_j = \left(P_0 \mu(p,.)^{-1} P_0 \right)^{-1} \left\{ -P_0 \mu(p,.)^{-1} P_\nabla v_j + P_0 G_j \right\},$$

for $G_j \in L^2(\Omega)^3$ and $j = 1, 2, 3$. This implies that system (4.2) is equivalent to:

$$\begin{array}{l} \textit{Find } \widetilde{\Pi} = (P_\nabla v_1, P_\nabla v_2, P_\nabla v_3, u_1, u_2, u_3, \gamma) \in \mathcal{D}(\widetilde{\mathbb{E}}), \textit{ such that}: \\ \left(B(p,.) - \widetilde{\mathbb{E}} \right) \Pi = \widetilde{F}, \end{array} \tag{5.5}$$

where $\widetilde{F} \in (\text{grad}H^1(\Omega))^3 \times L^2(\Omega)^4$, $B(p,.) \in \mathcal{B}\left(L^2(\Omega)^{13}\right)$ is a bounded operator which is also holomorphic on D_0, and $\widetilde{\mathbb{E}}$ is defined by:

$$\widetilde{\mathbb{E}} = \begin{pmatrix} 0 & 0 & 0 & \nabla & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \nabla & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \nabla & 0 \\ \text{div}P_\nabla & 0 & 0 & 0 & 0 & 0 & \partial_1 \\ 0 & \text{div}P_\nabla & 0 & 0 & 0 & 0 & \partial_2 \\ 0 & 0 & \text{div}P_\nabla & 0 & 0 & 0 & \partial_3 \\ 0 & 0 & 0 & \partial_1 & \partial_2 & \partial_3 & 0 \end{pmatrix},$$

with domain $\mathcal{D}(\widetilde{\mathbb{E}}) = \mathcal{D}(\mathbb{E}) \cap (\text{grad}H^1(\Omega))^3 \times L^2(\Omega)^4$. From (4.8), the embedding of $\mathcal{D}(\widetilde{\mathbb{E}})$ into $L^2(\Omega)^{13}$ is compact. It yields the compactness of the resolvent of $\left(\widetilde{\mathbb{E}}, \mathcal{D}(\widetilde{\mathbb{E}})\right)$. Since this operator is maximal dissipative and $B(p,.)$ is a bounded multiplicative operator of $L^2(\Omega)^{13}$ for all $p \in D_0$, solving (5.5) is the same as inverting a holomorphic family of closed operators with compact resolvent which can be done with Fredholm analytical theory. The end of the proof is identical to the one of theorem 4.1. □

As for theorem 4.1 we can here extend the previous results with:

ASSUMPTIONS 11 (For non-constant and tensorial elastic materials). *Suppose that $(E1)-(E2)-(E3)$ and $(E4)$ hold, with the additional condition*

(E4) There exists $\alpha > 0$ and $a(p,x) \in \mathbb{C}$, Lipschitz continuous for $x \in \overline{\Omega}$ and holomorphic for $p \in D_0$, satisfying $a(p,.)_{|\partial\Omega} = 1$, such that

$$\mathcal{R}e\left(< \mu(p,x)^{-1} a(p,x) X, \overline{X} >\right) \geq \alpha |X|^2,$$

for all $p \in D_0$, for almost all $x \in \Omega$ and for all $X \in \mathbb{C}^3$.

COROLLARY 5.2. *If assumption 11 is verified, then equation (5.2) is well-posed for all $p \in D_0 \backslash S$ where $S \subset D_0$ is a discrete, locally finite and possibly empty set of D_0. Moreover, the solution is continuous with respect to the data and the application $p \in D_0 \backslash S \longmapsto (v(p,.), u(p,.), \gamma(p)) \in L^2(\Omega)^{13}$ is holomorphic.*

Proof. We follow the proof of corollary 4.5. Perform the changes of unknown $X_j(p) = a(p,.)v_j$ and $\gamma(p) = a(p,.)\widetilde{\gamma}$ and then project the system according to Hodge decomposition (4.3). Thus, from $(E4)$, it comes:

$$P_0 X_j = \left(P_0 \mu(p,.)^{-1} a(p,.) P_0\right)^{-1} \left\{-P_0 \mu(p,.)^{-1} a(p,.) P_\nabla X_j + P_0 G_j\right\},$$

for $j = 1,2,3$. This implies that system (4.2) is equivalent to:

$$\begin{array}{l} \textit{Find } \widetilde{\Pi} = (P_\nabla X_1, P_\nabla X_2, P_\nabla X_3, u_1, u_2, u_3, \widetilde{\gamma}) \in \mathcal{D}(\widetilde{\mathbb{E}}), \textit{ such that}: \\ \left(B(p,.) - \widetilde{\mathbb{E}}\right) \Pi = \widetilde{F}, \end{array}$$

where $F \in (\text{grad}H^1(\Omega))^3 \times L^2(\Omega)^4$, $B(p,.) \in \mathcal{B}\left(L^2(\Omega)^{13}\right)$ is a bounded operator which is also holomorphic on D_0. The end of the proof is the same as for theorem 5.1. □

6. Study of some examples. This section is dedicated to illustrate our results with some examples from the literature. We successively apply them to the study of Maxwell's equations with a periodical array of SRR, a Drude-Born-Fedorov system with some chiral metamaterial made from the Ω-particle resonator model or with a bi-anisotropic metamaterials. Examples that are also considered are the wave equation with some absorbing boundary condition of Perfectly Matched Layers (PML) type, and the wave equation with a homogenized acoustic metamaterial having negative bulk modulus.

6.1. Periodic array of Split-Ring-Resonator (SRR). The Split Ring Resonator (SRR) have been introduced by J.B. Pendry in 2000 as the first example of negative index material [30, 33]. Some studies dealing with a periodic array of S.R.R have followed [18, 29, 32] but the well-posedness of this system remains unanswered at the best of our knowledge. The effective parameters involved in (3.1) of a periodical array of interspaced conducting non-magnetic Split-Ring-Resonators and continuous wires calculated in [32] have the following expressions:

$$\begin{cases} \varepsilon(p,x) = \left(1 + \frac{w_G^2}{p^2}\right)\mathbb{I}_3, \\ \mu(p,x) = \left(1 + \frac{Fp^2}{-p^2 - w_0^2 + p\Gamma}\right)\mathbb{I}_3, \\ \beta(p,x) = 0, \end{cases} \tag{6.1}$$

where $w_G > 0$ is the plasma pulsation of gold, $w_0 = \sqrt{\frac{3l}{\pi^2\mu_0 C r^3}}$ and $\Gamma = \frac{2l\rho}{r\mu_0} > 0$ are constants. Here ρ is the resistance per unit length of the rings measured around the circumference, l is the distance between layers, a is the lattice parameter, r is a geometrical parameter (defined on figure 1 of [32]) and C is the capacitance associated with the gaps between the rings.

We consider here a homogenized SRR embedded in a connected bounded open set $\Omega \subset \mathbb{R}^3$ with connected $\mathcal{C}^1$ boundary. We invoke theorem 3.2 to study (3.1)-(6.1). So we have to check assumption 3.

(B1): ε and μ defined by (6.1) behave on p as rational fractions and thus are holomorphic on $D_0 = \mathbb{C}\backslash Z$, where Z is the set of zeros of their denominators. Namely $Z = \left\{(\Gamma - \sqrt{\Gamma^2 - 4w_0^2})/2, 0, (\Gamma + \sqrt{\Gamma^2 - 4w_0^2})/2\right\}$ where $\sqrt{y}$ is equal to $i\sqrt{-y}$ when $y < 0$.

(B2): Coefficients ε and μ are constant in x and thus have the requested regularity.

(B3): Let $g : p \in \mathbb{R}\backslash\{0\} \longmapsto -p^2 - w_0^2 + p\Gamma \in \mathbb{R}$. The maximum of g is reached for $p_0 = \frac{\Gamma}{2}$. As $g(p_0) > 0$, (B3) is satisfied.

Hence, theorem 3.2 implies the well-posedness of the Maxwell's equations in presence of homogenized SRR (3.1)-(6.1), with $f \in H(\mathrm{div}, \Omega)^2$, for all p in $D_0\backslash\widetilde{S}$ where $\widetilde{S}$ is a discrete, locally finite and possibly empty set of D_0.

6.2. A bi-anisotropic metamaterial. We consider now a material described by a lattice composed of bi-anisotropic homogenized Split-Ring-Resonator as studied in [21]. The physical parameters of the homogenized material are now defined by (3.5) with the following parameters:

$$\begin{cases} \varepsilon(p) = 1 + (\frac{dc_0}{l^2})^2 \frac{F}{w_{LC}^2 + p^2 - p\gamma}, \\ \mu(p,.) = 1 - \frac{Fp^2}{w_{LC}^2 + p^2 - p\gamma}, \\ \xi(p) = i\frac{dc_0}{l^2}\frac{Fp}{w_{LC}^2 + p^2 - p\gamma}, \\ \zeta(p,.) = -i\frac{dc_0}{l^2}\frac{Fp}{w_{LC}^2 + p^2 - p\gamma}. \end{cases} \tag{6.2}$$

where l and d are positive constant, c_0 is the speed of light in the vacuum, F is the S.R.R. volume filling fraction, $\gamma > 0$ is the damping and $w_{LC} > 0$ is the LC eigenfrequency.

Theorem 3.2 will be used to study the system (3.2)-(6.2), and hence assumption 3 has to be checked.

(B1): Coefficients (6.2) depend on p like rational fractions. So, they are holomorphic on $\mathbb{C}\backslash Z$, with $Z = \left\{ \left(\gamma \pm \sqrt{\gamma^2 - w_{LC}^2}\right)/2 \right\}$.

(B2): $\varepsilon(p,x), \mu(p,x), \xi(p,x)$ and $\zeta(p,x)$ are obviously Lipschitz continuous in $x \in \overline{\Omega}$ for all $p \in \mathbb{C}\backslash Z$. To show that $\varepsilon(p,x)\mu(p,x) - \xi(p,x)\zeta(p,x)$ does not vanish in a well-suited domain $D_0 := \mathbb{C}\backslash(Z \cup \{p_i, i = 1 \cdots 5\})$, we compute the zeros p_i (which do not depend on x) in p of this function using the explicit values of $dc_0/l^2 = 0.75 w_{LC},\ F = 0.3$ and $\gamma = 0.05 w_{LC}$ (see figure 2 of [21]):

$$\begin{cases} p_1 = (0.1522392099 + 1.128302032i)w_{LC}, \\ p_2 = (-0.09152492420 + 1.131232555i)w_{LC}, \\ p_3 = (-0.09152492420 - 131232555i)w_{LC}, \\ p_4 = (0.1522392099 + 1.128302032i)w_{LC}, \\ p_5 = 0. \end{cases}$$

(B3): To fulfil $(B3)$ one needs to find a $p_0 \in D_0$ satisfying the constraint given in (B3). It can be done by looking for a p_0 such that $K(p)$ (3.5) is coercive at this point. Its spectrum is

$$\sigma(K(p,.)) = \left\{ \left(\varepsilon(p) + \mu(p,.) - \sqrt{(\varepsilon(p)^2 - 2\mu(p,.)\varepsilon(p) + \mu(p,.)^2 + 4\xi(p)^2)}\right)\frac{p}{2}, \right.$$
$$\left. \left(\varepsilon(p) + \mu(p,.) + \sqrt{(\varepsilon(p)^2 - 2\mu(p,.)\varepsilon(p) + \mu(p,.)^2 + 4\xi(p)^2)}\right)\frac{p}{2} \right\}.$$

For $p = w_{LC}$ we have $K(w_{LC}) = K(w_{LC})^*$ and

$$\sigma(K(w_{LC})) = \{0.7997334285 w_{LC},\ 1.132958880 w_{LC}\},$$

both with multiplicity 3. Finally $K(w_{LC})$ is coercive.

Consequently, theorem 3.2 can be applied to show that the Maxwell's equation (3.2) in presence of a bi-anisotropic material described by a homogenized S.R.R. 6.2) is well-posed with right member $f \in (H(\mathrm{div}, \Omega))^2$ for all p in $D_0\backslash S$ where S in an exceptional set of values.

6.3. Chiral metamaterial based on the Ω-particle resonator model. We consider a material described by a Ω-particle resonator model (see figure 6 [35]) embedded in a connected bounded open set $\widetilde{\Omega}$ (the notations changes here to avoid confusion) which is filled with a material whose positive parameters are denoted $\varepsilon_b,\ \mu_b$. The electromagnetic properties are described by equation (3.3) with the following parameters:

$$\begin{cases} \varepsilon(p,x) = \varepsilon_b + \frac{\Omega_\varepsilon w_0^2}{w_0^2 + p^2 - p\gamma}\mathbb{I}_3, \\ \mu(p,x) = \mu_b - \frac{\Omega_\mu p^2}{w_0^2 + p^2 - p\gamma}\mathbb{I}_3, \\ \beta(p,x) = \frac{\Omega_\beta p}{i\left(w_0^2 + p^2 - p\gamma\right)}\mathbb{I}_3, \end{cases} \tag{6.3}$$

where w_0, γ and $\Omega_{\varepsilon,\mu,\beta}$ are some positive constants defined by the homogenized model [35].

To be used on system (3.3)-(6.3), theorem 3.2 now requires to check assumption 4:

(C1): The permittivity, the permeability and the chirality defined by (6.3) are rational fractions of p. Hence $(C1)$ is satisfied on $\mathbb{C}\backslash Z$ where $Z = \{(\gamma \pm \sqrt{\gamma^2 - w_0^2}\,)/2\}$ is the set of zeros of their denominators.

(C2): ε, μ and β do not depend on x, hence the required regularity is obviously satisfied. The quantity $p^2\varepsilon(p,x)\mu(p,x)M(p,x)$ does not vanish since $p \notin Z \cup \{p_j, j = 1 \cdots 5\}$, where $(p_j)_j$ are the zeros of this function. Using the explicit values of $\Omega_{\varepsilon,\mu,\beta}$ and ε_b, μ_b (see p.12 of [35]) we find

$$\begin{cases} p_1 = 0,05146743450 - 1.909033734i, \\ p_2 = 0,05146743450 + 1.909033734i, \\ p_3 = 0,05498575999 - 1.927011242i, \\ p_4 = 0,05498575999 + 1.927011242i, \\ p_5 = 0. \end{cases}$$

(C3): We use again the notation $K(p)$ (3.5) and infer its spectrum to be: $\sigma(K(p))$

$$= \left\{ \frac{\varepsilon(p) + \mu(p) - \sqrt{\varepsilon(p)^2 - 2\varepsilon(p)\mu(p) + \mu(p)^2 - 4\varepsilon(p)^2\mu(p)^2p^2\beta(p)^2}p}{2 + 2p^2\varepsilon(p)\mu(p)\beta(p)} \right.$$
$$\left. \frac{\varepsilon(p) + \mu(p) + \sqrt{\varepsilon(p)^2 - 2\varepsilon(p)\mu(p) + \mu(p)^2 - 4\varepsilon(p)^2\mu(p)^2)p^2\beta(p)^2}p}{2 + 2p^2\varepsilon(p)\mu(p)\beta(p)} \right\},$$

both with multiplicity 3. Thus, the spectrum of $K(1) = K(1)^*$ is $\sigma(K(1)) = \{0.9536351793, 3.444237487\}$, both with multiplicity 3, and then $K(p_0)$ is coercive for $p_0 = 1$.

Consequently, theorem 3.2 can be applied to show that the Maxwell's equation (3.3) in presence of a chiral material described by a Ω-particle resonator model (6.3) is well-posed with right member $f \in \left(H(\mathrm{div}, \widetilde{\Omega})\right)^2$ for all p in $D_0\backslash S$ where S is an exceptional set of values.

6.4. Acoustic metamaterial with negative modulus. We consider now a homogenized acoustic metamaterial made from short tubes with side hole used as unit cell like the one studied in [23]. The homogenized media has negative modulus at some frequencies. It has been remarked in [23] that such material behaves analogously to the one having negative permittivity. The physical modelling is given by system (4.1) with parameters

$$\Gamma(p,x) = p\vartheta\mathbb{I}_3, \quad n(p,x) = pB^{-1}\left(1 - \frac{w_{sh}^2}{\gamma p - p^2}\right), \tag{6.4}$$

where $x \in \Omega$, γ is the damping term, $\vartheta = 1.21\ kg/m^3$, $B = 1.42 \times 10^5\ Pa$ is the bulk modulus of air and $w_{sh} > 0$ is defined in [23].

According to theorem 4.1 we are going to check assumption 8.

(A1): Coefficients in (6.4) are meromorphic with pole at 0 and γ, hence they are holomorphic in $D_0 = \mathbb{C}\backslash\{0,\gamma\}$.

(A2): Γ and n are not depending on x, hence they have the required regularity and invertiblity since they are defined for any x when p belongs to D_0.

(A3): For $p_0 \in \mathbb{R}^+$ large enough it comes $p_0B^{-1}\left(1 - \frac{w_{sh}^2}{\gamma p_0 - p_0^2}\right) > 0$, thus satisfying the condition.

Finally, theorem 4.1 can be applied to show that equation (4.1)-(6.4) is well-posed for all $p \in D_0\backslash S$ where S is a discrete, locally finite and possibly empty set of D_0.

6.5. Approximate cloaking for the Wave equation. It is well known that perfect acoustic or electromagnetic cloaking is hard since it uses singular transformations optics which lead to singular material parameters (see [14] and references therein). This difficulty has been overcome recently with the *approximate* cloaking (see [15] and references therein) giving rise to non-singular parameters which are bounded from above and below. However, much of *approximate* cloaking devices do not use materials with frequency dependent parameters such as metamaterials. Nevertheless, there exists theoretical materials defined as complex absorbing boundary conditions [27] which perform *approximate* cloaking since they absorb strongly the scattered waves. We thus consider them as metamaterial in a broader sense. Actually, as an application of our theory, we are going to retrieve that the wave equation (4.1) in presence of complex absorbing boundary condition of Perfectly-Matched-Layers (PML) type is well-posed.

Now we introduce the construction of PML via a complex stretching of $\mathbb{R}^3$ [27]. Let Θ be a convex subset of Ω with $\mathcal{C}^3$ boundary whose outward unitary normal is noted $\mathbf{n}_\Theta$. Since Θ is convex, there exists, for all $x \in \Omega$, a unique $\eta_x \in \partial\Theta$ such that $h = dist(x, \partial\Theta) = |x - \eta_x|$ and $x = \eta_x + h\mathbf{n}_\Theta(\eta_x)$. Let $\sigma \in C^3(\mathbb{R}^+, \mathbb{R}^+)$ such that $\lim_{s\longrightarrow+\infty} \sigma(s) = \lim_{s\longrightarrow+\infty} \sigma^{'}(s) = +\infty$ and $\sigma(0) = \sigma^{'}(0_+) = 0$. A complex formulation of the PML is then obtained introducing the new coordinates:

$$\widetilde{x} : \mathbb{R}^3 \longrightarrow \mathbb{C}^3,\ x \longmapsto \widetilde{x}(x) = x + \frac{\sigma(h)\mathbf{n}_\Theta(\eta_x)}{p}. \tag{6.5}$$

At last, noting $J(p, x) = \nabla\widetilde{x}(p, x)$ the Jacobian of this transformation, we assume that (6.5 defines a bijective mapping for $(p, x) \in \mathbb{C}\backslash\{0\} \times \Omega$. The wave equation with absorbing boundary condition of PML type is then equation (4.1) with coefficients

$$\Gamma(p, x) = \sqrt{det(g(p, x))}\, g(p, x)^{-1}, \quad n(p, x) = p^2 det(g(p, x)), \tag{6.6}$$

where $g(p, x) = (J(p, x)^T J(p, x))^{-1}$.

We show that (4.1)-6.6) is well-posed with corollary 4.5, by checking assumption 9.

- (A1): For almost all $x \in \Omega$, the application $p \in D_0 = \mathbb{C}\backslash\{0\} \longmapsto g(p, x) \in Hom(\mathbb{C}^3)$ is holomorphic, and so are $p \mapsto \Gamma(p, .)$ and $p \mapsto n(p, .)$.
- (A2): For any $p \in D_0 := \mathbb{C}\backslash\{0\}$, $g(p, .)$ is invertible. Moreover, $g(p, .)$ and $g(p, .)^{-1}$ are bounded on Ω since the application σ belongs to $\mathcal{C}^3(\mathbb{R}^+, \mathbb{R}^+)$ and never vanish and since $x \mapsto \widetilde{x}(x)$ (6.5) is bijective by hypothesis.
- (A3): The tensor $g(p_0)$ is coercive with $p_0 = 1$. Hence, $(X, z) \mapsto |z|^2\mathcal{R}e(n(1, x)) + \mathcal{R}e\left\{< \Gamma(1, x)X, \overline{X} >\right\}$ is coercive too.
- (A4): Pose $a(p, x) = 1/\sqrt{det(g(p, x))}$, then $\left\langle \Gamma(p, x)a(p, x)X, \overline{X} \right\rangle = |J(p, x)X|^2$ which is coercive.

Hence, the system (4.1)-(6.6) is well-posed for all $p \in D_0\backslash S$ where S is a discrete, locally finite and possibly empty set of $D_0 = \mathbb{C}\backslash\{0\}$.

Remark 6.1. *Remark, in the same way, that we can prove that Maxwell's equations with absorbing boundary condition of PML type are well-posed too. Indeed the presence of convex PML implies that the physical parameters (see [27] for computations) in equation (3.1) are modified as follows*

$$\varepsilon(p, x) = \mu(p, x) = g(p, x)/\sqrt{\det(g(p, x))}. \tag{6.7}$$

The same study can now be led using theorem 3.6. Thus, we have to check assumption 5 where $(BT1) - (BT2) - (BT3)$ *are already proved with* $(A1) - (A2) - (A3)$.

The last item, (BT4), is satisfied taking $a_\varepsilon(p,x) = a_\mu(p,x) = \sqrt{\det(g(p,x))}$. Hence, the Maxwell's equations (3.1)-(6.7) with PML is well-posed for all $p \in D_0\backslash S$ where S is a discrete, locally finite and maybe empty set of $D_0 = \mathbb{C}\backslash\{0\}$.

6.6. Elastic metamaterial. The last example to be addressed in this paper concerns a model introduced in [36]. It consists of a three phase composite with coated spheres with radius r_j embedded in a host material represented by a connected bounded open set Ω. Each region of the doubly-coated sphere is assumed to be elastic material characterized by mass density ρ_i, Lamé coefficients λ_i and μ_i with the subscript $i = 1, 2, 3$ representing separately the sphere, the coating and the host. It is shown in [36] that this composite has negative effective mass density ρ_{eff}, negative μ_{eff} and negative bulk modulus κ_{eff}. We show, using theorem 5.1, that the elasticity system (5.1) in presence of this material is well-posed. The effective Lamé coefficients of the media are defined by:

$$\begin{cases} \mu_{eff}(p) = \mu_3 + \\ \quad \dfrac{(\mu_1-\mu_2)r_1^2\left(u'_{r,2}(r_1)+3u'_{\theta,2}(r_1)\right)+(\mu_2-\mu_3)r_1^2\left(u'_{r,2}(r_2)+3u'_{\theta,2}(r_2)\right)}{r_3^2\left(u'_{r,2}(r_3)+3u'_{\theta,2}(r_3)\right)}, \\ \lambda_{eff}(p) = \kappa_{eff}(p) - \dfrac{2}{3}\mu_{eff}(p), \\ \kappa_{eff}(p) = \kappa_3 + \\ \quad \dfrac{r_1(\kappa_1-\kappa_2)E_0^{13}(s,r_1)c_0^{(1)}+r_2(\kappa_2-\kappa_3)\left(E_0^{11}(h,r_2)a_0^{(3)}+E_0^{13}(h,r_2)\right)}{r_3\left(E_0^{11}(h,r_3)a_0^{(3)}+E_0^{13}(h,r_3)\right)}, \end{cases} \tag{6.8}$$

where E_{ij}^k and $u'_{r,\theta,l}$ depend on p and invoke spherical Bessel and Hankel functions of the first kind (see the appendix of [36]). We check assumption 10.

- (E1): The holomorphy of $\mu_{eff}(p)$ and $\lambda_{eff}(p)$ follows from the holomorphy of spherical Hankel and Bessel functions of the first kind (see [26] for definitions and properties of these special functions). Hence the Lamé coefficients (6.8) are holomorphic on $D_0 = \mathbb{C}\backslash\{0\}$.
- (E2): The Lamé coefficients (6.8) do not depend on $x \in \Omega$ giving trivialy the condition.
- (E3): According to [36] figures 4 and 6, for $p_0 = 1$ the coefficients μ_{eff} and λ_{eff} are strictly positive constants.

Thus, theorem 5.1 shows that for all $p \in D_0\backslash S$, where S is a set of exceptional values, system (5.2)-(5.4)-(6.8) is well-posed.

7. Conclusion and remarks. In this paper we have studied the well-posedness of some linear partial differential equations coming from the modelling of electromagnetics, acoustics and elastodynamics phenomenons in metamaterials. We have shown some generic well-posedness for each of the previously mentioned systems in presence of metamaterials, under assumptions relevant for some models from the literature. Moreover, we have successively applied our results on a periodical array of Split-Ring-Resonator for Maxwell's equations, a chiral metamaterial built from the Ω-particle resonator model, a bi-anisotropic metamaterial, some absorbing boundary conditions of PML type for the wave equation and a homogenized acoustic metamaterial having negative bulk modulus. We have also examined some elastic metamaterials by introducing a elasticity system for which a well-posedness result have been demonstrated.

However, some remarks have to be formulated. Concerning our results, we do not show that a particular model is well-posed since we discard a discrete, locally finite and possibly empty set of frequencies. Fortunately the frequencies for which the problem is ill-posed are isolated and the solution is holomorphic. Thus any small variation of p would make it well-posed. On the other hand, we are not able to provide explicit values for the singular frequencies for which the problem is not well-posed since we work with domain having not any specific shape. Nevertheless, we provide in this paper sufficient conditions on the material to ensure discreteness and local finiteness of singular frequencies.

Remark also that we require smoothness in the spatial variable (at least Lipschitz continuous) of the multiplicative operators involved in equation we have studied here thought the assumptions of our theorems. Consequently, we cannot study transmission problems between "classical" materials and metamaterials, as it is done in [1, 3], since the multiplicative operator $K(p,.)$ is (only) $L^\infty(\overline{\Omega})$.

Acknowledgement. The authors want to thanks the anonymous referee for useful comments, references and suggestions to improve the overall presentation of this paper.

REFERENCES

[1] A.S. Bonnet-BenDhia, Jr. P. Ciarlet, C.M. Zwölf, Time harmonic wave diffraction problems in materials with sign-shifting coefficients, J. Comput. Appl. Math, vol. 234, number 6, p. 1912-1919, 2010.

[2] A.S. Bonnet-BenDhia, M. Dauge, K. Ramdani, Analyse spectrale et singularités d'un problème de transmission non coercif, Comptes-Rendus àl'Académie des Sciences Série I, vol. 328, number 8, p. 717-720, 1999.

[3] A.-S. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet Jr., Optimality of T-coercivity for scalar interface problems between dielectrics and metamaterials, http://hal.archives-ouvertes.fr/hal-00564312/, 2011.

[4] M. Cessenat, Mathematical Methods in Electromagnetism: Linear theory and applications, World Scientific Singapore and River Edge, 1996.

[5] H. Chena, C.T. Chan, Acoustic cloaking in three dimensions using acoustic metamaterials, Applied Physics Letters, vol. 91, issue 18, p. 183518, 2007.

[6] M. Costabel, E. Stephan, Integral equations for transmission problems in elasticity, Journal of Integral Equations and Applications, vol. 2, p. 211-223, 1990.

[7] P. Courilleau, T.H. Molinaro, I.G. Stratis, On the controllability of time-harmonic electromagnetic fields in chiral media, Advances in mathematical sciences and applications, vol. 16, number 2, p. 491-502, 2006.

[8] S.A. Cummer, D. Schurig, One path to acoustic cloaking, New Journal of Physics, vol. 9, p. 45, 2007.

[9] R. Dautray, J.L. Lions, Analyse mathématiques et calcul numérique pour les sciences et les techniques (T.5), Masson, 1988.

[10] Y. Ding, Z. Liu, C. Qiu, J. Shi, Metamaterial with Simultaneously Negative Bulk Modulus and Mass Density, Physical review letter, vol. 99, number 9, p. 93904, 2007.

[11] D. Felbacq, G. Bouchitté, Left-Handed media and homogenization of photonic crystals, Optics letters, vol. 30, number 10, p.1189–1191 2004.

[12] P. Fernandes, M. Raffetto, Well posedness and finite element approximability of time-harmonic electromagnetic boundary value problems involving bianisotropic materials and metamaterials, Mathematical Models and Methods in Applied Sciences, vol. 19, number 12, p. 2299-2335, 2009.

[13] K.O. Friedrich, Differential forms on Riemannian manifolds, Communications on Pure and Applied Mathematics, vol.8, number 4, p. 551-590, 1955.

[14] A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann, Cloaking Devices, Electromagnetic Wormholes and Transformation Optics, SIAM Review, vol. 51, p. 3-33, 2009.

[15] A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann, Isotropic transformation optics: approximate acoustic and quantum cloaking, New Journal of Physics, vol. 10, p. 115024, 2008.

[16] S. Guenneau, A.B. Movchan, G. Petursson, S.A. Ramakrishna, Acoustic meta-materials for sound focussing and confinement, New Journal of Physics, vol. 9, p. 399, 2007

[17] S. Guenneau, F. Zolla, Homogenization of 3D finite chiral photonic crystals, ScienceDirect, Physica B, vol. 394, p. 145-147, 2007.

[18] B. Kanté, S.N. Burokur, F. Gadot, A. De Lustrac, Métamatériaux à indices de réfraction négatif en infrarouge, UMR 8622, 91405 Orsay, F-91405.

[19] T. Kato, Perturbation theory for linear operators, Classics in mathematics Springer, 2nd edition, 1995.

[20] R.V. Kohn, H. Shen, M.S. Vogelius and M.I. Weinstein, Cloaking via change of variables in electric impedance tomography, Inverse problems, vol. 24, p. 015016, 2008.

[21] C.E. Kriegler, M.S. Rill, S. Linden, M. Wegener, Bianisotropic photonic metamaterials, IEEE Selected Topics in Quantum Electronics, vol. 16, number 2, p. 367-375, 2010.

[22] M. Lassas, The essential spectrum of non-selfadjoint Maxwell operator in a bounded domain, Journal of Mathematical Analysis and Applications, vol. 224, p. 201-217, 1998.

[23] S.H. Lee, C.M. Park, Y.M. Seo, Z.G. Wang, C.K. Kim, Acoustic metamaterial with negative modulus, Journal of Physics: Condensed Matter, vol. 21, p. 175704, 2009.

[24] A.J. Majda, Coercive inequalities for nonelliptic symmetric systems, Comm. pure Appl. Math., vol. 28, p. 1119-1133, 1975.

[25] G.W. Milton, P. Seppecher, G. Bouchitté, Minimization variational principles for acoustics, elastodynamics and electromagnetism in lossy inhomogeneous bodies at fixed frequency, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science, vol. 465, number 2102, p. 367-396, 2009.

[26] M. Abramowitz, I.A. Stegun, Handbook of mathematical functions with formulas, graphs, and mathematical tables, Dover publications, 1964.

[27] M. Lassas, J. Liukkonen, E. Somersalo, Complex riemanninan metric and absorbing boundary conditions, J. Math. Pures Appl., vol.7, number 80, p. 739-768, 2001.

[28] S. Lin, J.G. Fleming, D.L. Hetherington, B.K. Smith, R. Biswas, K.M. Ho, M.M. Sigalas, W. Zubrzycki, S.R. Kurtz, J. Bur, A three-dimensional photonic crystal operating at infrared wavelengths, Nature, vol. 394, number 6690, p. 251–253, 1998.

[29] J.B. Pendry, Intense Focusing Of Light Using Metals, NATO ASI series ed. CM Soukoulis, 2000.

[30] J.B. Pendry, A.J. Holden, D.J. Robbins, W.J. Stewart, Magnetism from Conductors, and Enhanced Non-Linear Phenomena, IEEE transactions on microwave theory and techniques, vol. 47, 1999.

[31] J. Rauch, Symmetric positive systems with boundary characteristic of constant multiplicity, Transaction of the American Mathematical society, vo. 291, number 1, 1985.

[32] D.R. Smith, W.J. Padilla, D.C. Vier, S.C. Nemat-Nasser, S. Schultz, Composite Medium with Simultaneously Negative Permeability and Permittivity, physical review letters, vol. 84, number 18, p. 4184–4187, 2000.

[33] D.R. Smith, D. Schurig, J.B. Pendry, Negative refraction of modulated electromagnetic waves, Applied Physics Letters vol. 81, number 15, p. 2713, 2002.

[34] V.G. Veselago, The electrodynamics of substance with simultaneously negative values of ε and μ, Soviet Physics USPEKHI, vol. 10, number 4, p. 509–514, 1968.

[35] R. Zhao, T. Koschny, C.M. Soukoulis, Chiral metamaterials: retrieval of the effective parameters with and without substrate, Optics Express, vol. 18, number 14, p. 14553–14567, 2010.

[36] X. Zhou, G. Hu, Analytic model of elastic metamaterials with local resonances, Physical Review B, vol. 79, number 19, p. 195109, 2009.